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Portfolio Credit Risk Modelling. A Review of Two Approaches.

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Contents

1 Introduction

The modelling and management of credit risk is a core concern within banks and other lending institutions. Credit risk refers to the risk of losses due to some credit event as, for example, the default of a counterparty. Thus, credit risk is associated with the possibility that an event may lead to some negative effects which would not generally be expected and which are unwanted. The main difficulties, when modelling credit risk, arise from the fact that default events are quite rare and that they occur unexpectedly. When, however, default events take place, they often lead to significant losses, the size of which is not known before default. Although default events occur very rarely, credit risk is, by definition, inherent in any payment obligation. Complex underlying dependence structure of the obliger from a single portfolio can cause severe losses both due to industry (systematic) or even individual (idiosyncratic) shocks. Modern society relies on the smooth functioning of the banking and insurance systems and has a collective interest in the stability of such systems. This implies that not only the banks themselves, but also global and local level supervisors intervene and make efforts to develop risk modelling and optimize financial sector in the sense of a socially joint understanding of sustainable development. One particular regulatory issue, called the credit risk capital, is the central topic of this paper.

The main question is:“ How much credit risk capital a financial institution should put aside in order to overcome possible sector, country or even worldwide macroeconomical difficulties and meet its obligations to investors and supervisors?” Implicitly, this questions contains all mathematical building blocks needed to address the problem:

(i) model for credit risk,

(ii) a risk measure,

(iii) an algorithm to carry out the calculations.

There are many methodologies presented in the literature, all aimed at calculating one and only value - risk capital of a credit portfolio, yet the choice of a credit risk model and an appropriate risk measure is still an issue. In order to encourage convergence towards common standards and approaches in financial sector as such and, in particular, credit risk management, the Basel Committee on Banking Supervision (BCBS) frames voluntary guidelines for reasonable credit risk modelling. BCBS latest global regulatory standard is summarized in the, so called, Basel III, [5], and its partial review [6], which is an extension of Basel II, [4], and was developed in response to the deficiencies in the regulation, revealed during the late 2000s financial crisis. This thesis concentrates on two papers which reflect widely used in the industry credit risk capital calculation schemes and also fit the BCBS regulatory framework. The first is the paper by Cespedes et al. (2004). Its revised version in 2006, [1], is chosen for this work. The other paper is of Düllmann and Maschelein(2006),[3].

At the core of [3] we find the methodology presented by Michael Pykhtin (2004), [8].

The aim of this thesis is to revisit and compare [1] and [3]. We want to study possible practical limitations, drawbacks and outline convenient application frameworks to which one or another method fits better.

The goal of both papers is to calculate credit risk capital of a credit portfolio, defined in terms of a risk measure. [1] suggest to derive a scaling factor which relates full loss model to its simplified version. Credit risk capital for the simplified loss model can be calculated analytically. As discussed in section 4., derivation of the scaling factor relies on Monte-Carlo simulations. In contrast to that, [3] offers an analytical approximation result, which is based on the Taylor expansion of the risk measure of the input portfolio around the same portfolio composition under a simplified loss model and then adjusted by the first two expansion terms. The simplified loss model is similar to the one used in [1], thus also analytically tractable.

We give a short introduction to the credit risk in general in section 2 and list several widely used risk measures. In addition we introduce portfolio concentration risks, which are also imposed by supervisory authorities to be identified and measured. In section 3 we introduce different credit loss models, used in [1] and [3]. Sections 4 and 5 are devoted to the two different credit risk capital estimation methodologies, taking into account the impact of the concentration risk. A test run of both methodologies and their performance comparison is discussed in section 6. [1] methodology relies on Monte-Carlo simulations, which are not only time consuming but also introduce estimation variance, especially when dealing with small probability credit default events. This motivates for variance reduction techniques, which are discussed in section 7. Variance reduction can be considered as an alternative to the brute-force approach, which in context of this thesis means an increase in the number of Monte-Carlo simulations in a straightforward manner in order to obtain higher numerical precision. Section 8 concludes and summarizes the findings that were learned throughout the thesis.

All remaining errors are of the author of the thesis.

2 Modelling credit risk

We begin with a description of a general credit portfolio that will be considered throughout the whole paper. In the following subsection 2.1. we discuss different ways to measure default risks. At the end of this section we introduce the implicit portfolio risks, which affect the cross-loan dependence and thus contribute to the portfolio diversification issues.

Section 2 is based on discussion from Chapter 1 and 2 of the book by Lütkebohmert, [10].

2.1 General framework

Assume a credit portfolio consisting of N different borrowers with a single loan per borrower. Each loan $n = 1, 2, \ldots, N$ is assigned to one sector $k = 1, 2, \ldots, K$. Sectors are usually being chosen in a way to represent either regional, geopolitical, industrial or any other important specifications and help to identify the underlying dependence structure among different loans. We focus on a Merton type one step (one year) credit risk model described via a credit portfolio random loss L defined as

$$\begin{aligned} L &= \sum_{n=1}^{N} \mathrm{EAD}_n \cdot \mathrm{LGD}_n \cdot \mathbb{1}_{\{X_n \leq \Phi^{-1}(\mathrm{PD}_n)\}} \\ &= \sum_{k=1}^{K} \sum_{n \in \mathrm{Sector}\ k} \mathrm{EAD}_n \cdot \mathrm{LGD}_n \cdot \mathbb{1}_{\{X_n \leq \Phi^{-1}(\mathrm{PD}_n)\}} \end{aligned} \tag{1}$$

where

- EAD_n is borrower's n exposure at default expressed in monetary values.

If a borrower defaults it does not necessarily mean that the creditor receives nothing from him. There is a chance that the borrower will partly recover, meaning that the creditor might receive an uncertain fraction of the notional value of the claim. LGD is meant to capture this behaviour.

- LGD_n is borrower's n loss given default expressed as a ratio of the full loan size EAD_n,
- PD_n denotes (one year) default probability of borrower n,
- X_n can be interpreted as the well-being indicator of borrower n, assumed to have standard normal distribution.

Assume LGD_n, EAD_n and PD_n are nonnegative and deterministic for all n. Denote

$$D_n = \mathbb{1}_{\{X_n \leq \Phi^{-1}(\mathrm{PD}_n)\}}. \tag{2}$$

More formally, X_n describe asset log-returns (standardized) of the n^{th} borrower, assuming that they follow classical Black-Scholes model. Thus X_n is assumed to be standard normal. This simplifying assumption is borrowed from asset price modelling, where the classical geometrical Brownian motion (Black-Scholes model) was for a long time the central stock dynamics model. Since stock prices reflect company's well-being and all relevant market information, one can apply the same reasoning for the well-being or, differently said, creditworthiness of a borrower n. Apart from that, there are practical issues with data gathering, which is easy to interpret in a multivariate normal framework, e.g., sampling data for a specified covariance matrix. $\Phi^{-1}(\cdot)$ denotes the inverse of a cumulative distribution function (cdf) $\Phi(\cdot)$ of a standard normal distribution $\mathcal{N}(0,1)$. This type of model, in which we evaluate borrowers liabilities (well-being or functionality) via a threshold, is called a threshold model. Thus the occurrence of a default (over the following year) is assumed to take place if the functionality conditions of the borrower (firm) n meet some pre-defined unsatisfactory level $\Phi^{-1}(\mathrm{PD}_n)$. If this happens, the bank loses $\mathrm{EAD}_n \cdot \mathrm{LGD}_n$.

Assumption 2.1. *The exposure at default EAD_n, the loss given default variable LGD_n and the default indicator D_n,* (2)*, of any borrower n are independent.*

Note that the default indicators D_n and D_m of different borrowers n and m are not assumed to be independent. This is implied by occasionally tight connections across different businesses. Different firms may depend not only on the same global macroeconomic factors but also on the well-being of their business partners, particular industry sector. Thus a default of one firm may cause domino effect leading to financial difficulties or even defaults of their partners. Generally, a binary random vector of default indicators

$$\mathbf{D} = (D_1, D_2, \ldots, D_N) \tag{3}$$

can be defined, with the joint default probability function given by

$$p(\mathbf{d}) = \mathbb{P}(D_1 = d_1, D_2 = d_2, \ldots D_N = d_n) \tag{4}$$

for $\mathbf{d} \in \{0,1\}^N$ and the marginal default probabilities $\mathbb{P}(D_n = 1)$ for all n. We will partly account for this dependence via imposing joint factors across different X_n. Partly, because the dependence structure will rely on some industry sector performance indicators and not on a default of one or another particular borrower. See subsection 2.3. for related discussion. See next section for the details concerning X_n.

It is often the case that one is interested in the loss as the ratio of the total portfolio size. In that case one can slightly modify (1) by changing EAD to

$$w_n = \frac{EAD_n}{\sum_i EAD_i} \quad \text{for all } n. \tag{5}$$

2.2 Risk measures

It is in general impossible to precisely predict possible losses. Yet to some extent banks can insure themselves against possible shocks. One way of assessing default risks is to introduce a risk measure based on a portfolio loss distribution. These are typically statistical quantities describing the conditional or unconditional loss distribution of the portfolio over some predetermined time horizon.

Minimum capital requirements can be defined in many ways. For instance, a possible framework is to set the minimum risk capital equal to some percentile of the wighted sum of all assets. Another possibility is to consider the expected loss. Taking the expectation of (1) gives us

$$\mathbb{E}[L] = \sum_{k=1}^{K} \sum_{n \in \text{Sector } k} \text{EAD}_n \cdot \text{LGD}_n \cdot \text{PD}_n. \tag{6}$$

Another possibility is the so-called unexpected loss (UL) which is defined as the standard deviation of (1), thus

$$\text{U}L = \sqrt{\sum_{n=1}^{N} \sum_{m=1}^{N} \text{EAD}_n \text{EAD}_m \text{LGD}_n \text{LGD}_m \text{Corr}(D_n, D_m)}. \tag{7}$$

The main drawback of $\mathbb{E}[L]$ and UL is their inability to fully reflect extreme scenarios (which show to happen in practice, both on a regional and on a global level, e.g., financial crisis), which lie far in the right tail of the loss distribution (note that the amount of the loss is expressed in positive values).

One of the most widely used risk measures in the financial industry is the Value-at-Risk (VaR). It finds many applications in BCBS supervisory frameworks. Value-at-Risk describes the maximally possible loss which is not exceeded in a given time period with a given high probability, the so-called confidence level. A formal definition is the following.

Definition (VaR) 2.2. For a confidence level $q \in (0, 1)$, the Value-at-Risk (VaR) of a portfolio loss variable L at the confidence level q is defined as

$$\text{VaR}_q(L) = F_L^{\leftarrow}(q), \tag{8}$$

where $F_L^{\leftarrow}$ is the quantile of the cdf $F_L(x) = \mathbb{P}(L \le x)$ of L.

In general, VaR can be derived for different holding periods and different confidence levels. In credit risk management, however, the holding period is typically one year and typical values for q are 95%, 99% or 99.9%.

Note 2.3. It may be difficult to statistically estimate high level VaR due to the problem of numerically simulating such scenarios of extreme cases. This will be addressed later in the thesis.

Proposition 2.4. *For a normally distributed random variable $\zeta \sim \mathcal{N}(\mu, \sigma^2)$ it holds*

$$VaR_q(\zeta) = \mu + \sigma \Phi^{-1}(q) \tag{9}$$

for $q \in (0, 1)$.

Proposition 2.5. *For a deterministic monotonically decreasing function $g(x)$ and a standard normal random variable X the following relation holds*

$$VaR_q(g(X)) = g(VaR_{(1-q)}(X)) = g\left(\Phi^{-1}(1-q)\right). \tag{10}$$

There are two major drawbacks with VaR.

1. VaR is not a coherent risk measure since it is not subadditive.

Non-subadditivity means that, if we have two loss distributions F_{L_1} and F_{L_2} for two portfolios and if we denote the overall loss distribution of the merged portfolio $L = L_1 + L_2$ by F_L, then we do not necessarily have that

$$\mathrm{VaR}_q(F_L) \leq \mathrm{VaR}(F_{L_1}) + \mathrm{VaR}(F_{L_2}).$$

Hence, VaR of the merged portfolio is not necessarily bounded above by the sum of the VaRs of the individual portfolios which contradicts the intuition of diversification benefits associated with merging portfolios.

2. VaR gives no information about the size of the losses which occur with probability $1 - q$.

If the loss distribution is heavy tailed (and we will see this in examples of section 7), this can be a problem.

Both of the above mentioned drawbacks motivate for another risk measure, called the expected-shortfall (ES), also known as conditional VaR.

Definition (ES) 2.6. For a loss L with $\mathbb{E}[|L|] < \infty$ and distribution function F_L, the expected shortfall (ES) at confidence level $q \in (0,1)$ is defined as

$$\mathrm{ES}[L] = \mathbb{E}[L|L \geq \mathrm{VaR}_q(L)]. \tag{11}$$

Equivalently, (11) can be rewritten as

$$\mathrm{ES}[L] = \frac{1}{1-q}\int_q^1 \mathrm{VaR}_u(L)du. \tag{12}$$

Thus, ES is a weighted average VaR of the whole loss distribution tail above $\mathrm{VaR}_q(L)$ level. Rephrased, an expected loss that is incurred in the event that $\mathrm{VaR}_q(L)$ is exceeded. Definition 2.6. implies the inequality $\mathrm{ES}_q \geq \mathrm{VaR}_q$. ES is subadditive and contrary to VaR also captures the average value of the extreme loss (which may occur with probability $1 - q$). ES is also introduced in Basel III, [6], for market risks (although VaR is still kept in other risk frameworks) and is gaining popularity.

Figure 1 is an example visualizing all presented risk measures, assuming that the loss variable L is standard normal and the percentile level is fixed at $q = 95\%$.

It can happen that a bank loses all of its positions in a specified time period. But it is economically inefficient to protect yourself against such unlikely event

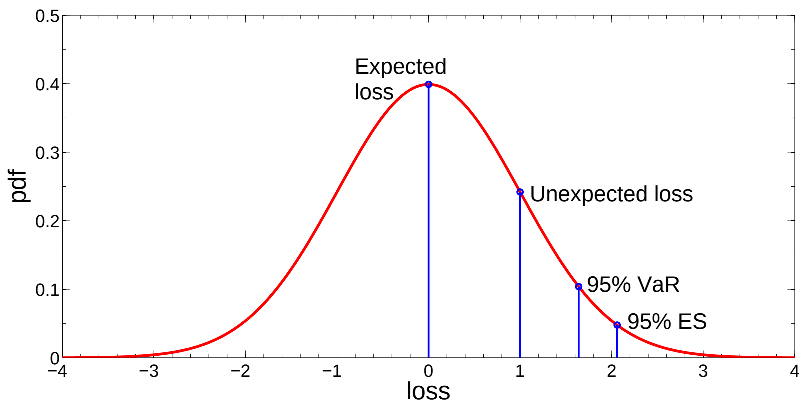

Figure 1: Probability density function of $\mathcal{N}(0,1)$ distributed loss variable and its expected loss, unexpected loss (standard deviation), VaR, (8), and ES, (11), at a 95% confidence level.

by holding some capital buffer. Banks are aimed towards profit, thus a tradeoff between minimizing its risk capital buffer (and still guarantee to some extent all deposit obligations) and investing has to be faced. In a period of growth one could actually use any of the previously presented risk measures in order to manage possible losses. This is not the case in periods of financial distress when the relative behaviour of different risk measures can vary extremely and therefore can be crucial for a company's ability to overcome this period. These issues are also addressed in Basel III, [5], context by asking financial institutions to calculate stressed risk measures, meaning that risk measures have to be calculated based also on the historical data of a financial distress in order to correct for the final risk capital.

2.3 Sector and Name concentration

In order to achieve as realistic as possible credit risk measurements it is important to realise the underlying dependence structure across different loans stemming either from the same sector (intra-sector dependence) or different sectors (cross- or inter-sector dependence). This is important not only from the sustainability point of view but also gives opportunities to adequately reduce risk capital buffers for the banks which propose higher diversity and thus increased quality of a credit portfolio. One can think of some basic credit risk capital requirement models, e.g., calculated as a share of the weighted portfolio loans

$$\text{risk\_capital} = 8\% \cdot \sum_{\text{loans}} \text{weight}_{\text{loan}} \cdot \text{loan\_size}$$

or defined via VaR for the asymptotical single risk factor (ASRF) model (see subsection 3.2), do not provide an additional option to quantify and account for the quality of a portfolio, thus partly ignoring the implicit dependence structure across different loans.

Definition (Concentration risk) 2.7. Concentration risks in credit portfolios arise from an unequal distribution of loans to single borrower (name concentration) or different industry or regional sectors (sector or country concentration). Moreover, certain dependencies as, for example, direct business links between different borrowers, or indirectly through credit risk mitigation, can increase the credit risk in a portfolio since the default of one borrower can cause the default of a dependent second borrower. This effect is called default contagion and is linked to both name and sector concentration.

Historical events have shown that ignorance of concentration risk led to serious difficulties in many financial institutions, particularly during crisis periods (see, for instance, [14]).

It may be difficult to account for all causal relations between different loans (thus firms) in a portfolio. Credit portfolios of many big banks can reach a size of more than 3000 liabilities. Hence it is important to quantify some unifying, leading variables to describe the dependence structure in a computationally feasible way. We split them into two groups:

- *systematic risk* factors which represent macroeconomical and industry-level changes and have influence on performance of each borrower (or actually reflect the collective performance of an industry sector on a regional or global level). Each borrower's sensitivity to this risk factor can be set individually.

- *idiosyncratic risk* factors which reflect each borrower's individuality.

In the introduction (section 1) a short insight in the methodologies of [1] and [3] was presented. The difference between the so-called simplified and the full loss model lies in the way of defining each borrower asset returns X_n, (1) and (2), whereas the portfolio itself (EADs, PDs and LGDs) stays unchanged. If all X_n are influenced by the same, single systematic risk factor, we call this a single-factor Merton type credit risk model (this was meant by the simplified model). In case there exist X_n and X_m for $n \neq m$, which are influenced by different systematic risk factors, the model is said to be multi-factor.

3 Merton type default model

We have already stated the general loss framework for a credit portfolio via (1). Whereas EADs, LGDs and PDs are assumed to be given, the well-being indicators X_n, (1) and (2), need to be modelled in order to incorporate the dependence structure, particularly the systematic and idiosyncratic risk factors. It is the topic of this section to introduce several ways of modelling X_n in the first part and to provide some analytical approximation results for risk measure calculations based on assymptotical analysis in the second part.

3.1 Multi-factor model

There are several ways to incorporate systematic and idiosyncratic risk factors in the model of X_n, thus to express borrowers performance in terms of these factors. We consider the set-up of (1).

Assumption 3.1. *The asset returns (or the well-being) X_n of borrower n are given by (after standardization[1])*

$$X_n = r_n Y_n + \sqrt{1 - r_n^2}\ \xi_n \quad \textit{for every } n \in \{1, 2, \ldots, N\} \tag{13}$$

where $Y_n, \xi_n \sim \mathcal{N}(0, 1)$ representing systematic and idiosyncratic risk parts respectively. The factor loadings r_n represent borrower's n sensitivity to systematic risk Y_n. r_n are chosen such that X_n stays standard normal. ξ_n is independent of Y_p, ξ_m for all $p \in \{1, 2, \ldots, N\}$ and $n \neq m$ respectively.

In context of this paper we consider following three models for Y_n. The first model introduces a single macro factor Z and a unifying systematic risk factor on a sector-level . Mathematically expressed,

$$Y_{k(n)} = \beta Z + \sqrt{1 - \beta^2}\ \eta_{k(n)}, \tag{14}$$

where $k(n) \in 1, 2, \ldots, K$ is the sector to which borrower n is assigned. Z is set to be standard normal, $\eta_k(n)$ are sector specific contributions to the systematic risk (in contrast to macro factor Z), all iid $\mathcal{N}(0, 1)$ and independent of Z.

Second model extends the previous one by allowing parameter β to be more generally an average factor correlation for a specific sector, thus

$$Y_{k(n)} = \beta_{k(n)} Z + \sqrt{1 - \beta_{k(n)}^2}\ \eta_{k(n)}. \tag{15}$$

Models (14) and (15) are used in [1]. A different, third approach is suggested by [8] and used in [3]. Let K original correlated systematic factors be decomposed into K independent standard normal systematic risk factors Z_k for $k \in \{1, 2, \ldots, K\}$.

$$Y_{k(n)} = \sum_{k'=1}^{K} \alpha_{k(n)k'} Z_{k'}, \tag{16}$$

[1] Normally distributed with zero mean and unit variance, also called as standard normal.

where the coefficients $\alpha_{k(n)k'}$ must satisfy $\sum_{k'=1}^{K} \alpha_{k(n)k'}^2 = 1$ to keep Y_n standard normal.

To relax the notation we write $\alpha_{nk'}$ instead of $\alpha_{k(n)k'}$ implicitly meaning that the vectors $(\alpha_{nk})_k, (\alpha_{mk})_k \in \mathbb{R}^K$ are equal if $k(n) = k(m)$, i.e., loans n and m represent the same sector.

Notice that the number of different Z_k in (16) corresponds to the number of sectors and this is not a coincidence. In practice, one usually searches for appropriate index-type instruments that can help to quantify performance of the underlying industry sectors. For the accounting purposes a financial institution may introduce more detailed sector definitions. But this makes no sense for the credit risk model in case different sector loans are set to be influenced by the same systematic risk factor because there is no financial data to make these loans sector-distinguishable. Thus often meaningful sector definitions stem from the available data framework.

It is in general impossible to directly calculate VaR or ES of a portfolio loss L (1), due to unknown F_L. Let us derive some first facts about L, that can provide some additional information about F_L.

The condition of the default indicator D_n, (2), $X_n \leq \Phi^{-1}(\mathrm{PD}_n)$ can be rewritten using (13) as $r_n Y_n + \sqrt{1-r_n^2}\, \xi_n \leq \Phi^{-1}(\mathrm{PD}_n)$, which is equivalent to

$$\xi_n \leq \frac{\Phi^{-1}(\mathrm{PD}_n) - r_n Y_n}{\sqrt{1-r_n^2}} \tag{17}$$

Having (17), we define the notion of a conditional probability of default $\mathrm{PD}_n(\cdot)$ as

$$\mathrm{PD}_n(Y_n) = \mathbb{E}[D_n | Y_n] = \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - r_n Y_n}{\sqrt{1-r_n^2}}\right). \tag{18}$$

Systematic risk factor term Y_n is the only random part in (18). Thus, for instance, using (16) we get conditional default probability for borrower n given realization $\mathbf{z} = (z_1, z_2, \ldots, z_K)$ of $(Z_1, Z_2, \ldots, Z_K)$ as

$$\mathrm{PD}_n(\mathbf{z}) = \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - r_n \sum_{k=1}^{K} \alpha_{nk} z_k}{\sqrt{1-r_n^2}}\right). \tag{19}$$

This gives rise to the conditional expectation of L, (1),

$$\mathbb{E}[L|\mathbf{z}] = \sum_{n=1}^{N} \mathrm{EAD}_n \cdot \mathrm{LGD}_n \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - r_n \sum_{k=1}^{K} \alpha_{nk} z_k}{\sqrt{1-r_n^2}}\right). \tag{20}$$

Since default probabilities depend only on Y_n (see (18)) we can compute joint distribution of the default indicator $\mathbf{D}$, (3), via integrating out Y_n terms as

$$\begin{aligned} &\mathbb{P}[D_1 = d_1, D_2 = d_2, \ldots, D_N = d_N] \\ &= \int_{\mathbb{R}^N} \prod_{n=1}^{N} \mathrm{PD}_n(y)^{d_n} (1 - \mathrm{PD}_n(y))^{1-d_n} dF_Y(\mathbf{y}), \end{aligned} \tag{21}$$

where F_Y denotes the cdf of the composite factors $(Y_1, Y_2, \ldots, Y_N)$ and $\mathbf{d} \in \{0,1\}^N$. Using substitution $q_n = \mathrm{PD}_n(\mathbf{y})$ we can rewrite (21) as

$$\begin{aligned} &\mathbb{P}[D_1 = d_1, D_2 = d_2, \ldots, D_N = d_N] \\ &= \int_{[0,1]^N} \prod_{n=1}^{N} q_n^{d_n} (1-q_n)^{1-d_n} dF(q_1, q_2, \ldots, q_N), \end{aligned} \tag{22}$$

where F is a cdf of a multivariate centred normal random vector with correlation matrix Γ, denoted as

$$F(q_1, q_2, \ldots, q_N) = \mathcal{N}_N(PD_1^{-1}(q_1), PD_2^{-1}(q_2), \ldots, PD_N^{-1}(q_N); \Gamma), \tag{23}$$

where $\mathcal{N}_N$ denotes an $N-$dimensional multivariate normal distribution with zero mean vector and correlation matrix Γ. An entry $\gamma_{mn} \in \Gamma$ is the correlation of X_m and X_n. In general, we can write the loss distribution as

$$\begin{aligned} &\mathbb{P}[L \leq l] = \\ &\sum_{\mathbf{d} \in \{0,1\}^N; \psi_n \leq l} \psi_n \cdot \mathbb{P}[D_1 = d_1, D_2 = d_2, \ldots, D_N = d_N], \end{aligned} \tag{24}$$

where $\psi_n = \psi_n(\mathbf{d}) = \mathrm{EAD}_n \cdot \mathrm{LGD}_n \cdot d_n$ for all n. Thus it is now a matter of how we choose the cdf F in (22).

Both in [1] and [3] authors work with the classical Gaussian copula when modelling dependence between systematic risk factors $\{Y_k\}_{k=1}^K$. Recall that a $d-$dimensional copula C is a distribution function on $[0,1]^d$ with standard uniform marginal distributions

$$C(u_1, u_2, \ldots, u_d) : [0,1]^d \to [0,1],$$

as by [19]. Copulas are used to describe the dependence structures across uniform random variables $U_1, U_2, \ldots, U_d$, which can be transformed into any random variables $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_d$ with cdf $F_1, F_2, \ldots, F_d$ by setting $\Upsilon_1 = F_1^{-1}(U_1)$, $\Upsilon_2 = F_2^{-1}(U_2), \ldots, \Upsilon_d = F_d^{-1}(U_d)$. The reason for using Gaussian copula in our framework of credit risk management is both the assumed underlying Black-Scholes asset dynamics model, giving rise to normally distributed log-returns, and also the economical interpretation of the systematic risk factors and their correlation matrix, which in practice equals the correlation matrix of different industry indices chosen by a bank as the systematic risk factors Z_k (recall (16)) for its portfolio model. We will see how to choose parameters α_{nk} from (16) in section 5.

Note that (24) together with (23) is sufficient to directly apply Monte-Carlo techniques for approximation purposes. Whereas [3] provides an analytical estimation for $\mathrm{VaR}(L)$ or $\mathrm{ES}(L)$, the methodology discussed in [1] is semi-analytical and relies on Monte-Carlo simulations. Typically one uses large Monte-Carlo simulations also for the reference result. Recall that risk measures as VaR or ES need a good tail estimation of the loss distribution, meaning that we are interested in rare events. The increasing computational time, that is needed for precise calculations in case of a heavy tailed L, motivates for importance sampling techniques, that we discuss in section 7.

3.2 Asymptotic single risk factor model (ASRF)

In this subsection we present first step towards analytical approximation of portfolios VaR. The so-called Asymptotic single risk factor (ASRF) model was introduced by Basel Committee incorporating the idea that a risk capital needed for a risky loan should not depend on the whole portfolio decomposition (called portfolio-invariance). One reason for that is fast and straightforward computation, as we will see. ASRF model also allows for a kind of comparison study across different companies and sectors. Yet neglecting portfolio decomposition is also its main drawback, since such an approach does not account for loan diversity, thus gives no information about how good (in the risk management sense) a loan fits some portfolio.

ASRF is based on the law of large numbers. When the number of loans tends to infinity, the idiosyncratic factors are diversified away and the only driving factor is the systematic risk. Such a portfolio is called infinitely fine grained.

As mentioned before, [1] and [3] start with a multi-factor model. They reduce it to a simplified one-factor credit risk model, calculate the risk capital under its framework and then adjust the result to account for the multi-factor case. Under ASRF model VaR can be calculated analytically and because of that ASRF model results are used as an analytical approximation for the risk capital under the simplified, one-factor model.

In the following two theorems and the related assumptions are presented. These theorems show how to calculate VaR in ASRF framework, thus provide an approximation for $\mathrm{VaR}_q(L)$ under a one-factor model.

Assumption 3.2. *Let the loan exposures fulfil the following conditions*

1. *Portfolios are infinitely fine grained, i.e., every single exposure contributes arbitrarily little to the total portfolio exposure.*
2. *Dependence across exposures is driven by a single systematic risk factor (i.e., $X_n = rY + \sqrt{1-r^2}\xi_n$ for all n).*

Assumption 3.3. *Assume that the variables $U_n = \mathrm{LGD}_n \cdot D_n \in [0,1]$, for $n = 1, 2, \ldots, N$, and are mutually independent conditionally on Y.*

The first condition of Assumption 3.2. is satisfied if the following Assumption 3.4. holds.

Assumption 3.4. *Let the loan exposure sizes fulfil the following conditions*

1. $\lim_{N\to\infty} \sum_{n=1}^{N} \mathrm{EAD}_n \to \infty$.
2. *$\exists \rho > 0$ such that the largest exposure share is of order $O(N^{-\frac{1}{2}+\rho})$. Thus the share of largest exposure shrinks to zero as the number of loans N increases.*

Theorem 3.5. *Under Assumptions 3.3. and 3.4. the strong law of large numbers implies*

$$L - \mathbb{E}[L|y] \xrightarrow{a.s.} 0 \text{ as } N \to \infty \tag{25}$$

where y is a realization of a single systematic risk factor Y.

This is the central result for the ASRF model. See [20, Prop 1] for a formal proof. Whereas in general Y can be a random vector in Theorem 3.5., it is no more the case in Theorem 3.6. and related Assumptions 3.6. See section 4 and 5 for the methods of switching from a multi-factor to a single-factor model.

Assumption 3.6. *There is an open interval B containing the q^{th} percentile $\mathrm{VaR}_q(Y)$ of the systematic risk factor Y and there is a real number $N_0 < \infty$ such that*

1. $\forall\ n,\ \mathbb{E}[U_n|y]$ *is continuous in $y \in B$,*
2. $\mathbb{E}[L|y]$ *is nondecreasing in $y \in B$ for all $N \geq N_0$, and*
3. $\inf_{y\in B} \mathbb{E}[L|y] \geq \sup_{y\leq \inf B} \mathbb{E}[L|y]$ *and* $\sup_{y\in B} \mathbb{E}[L|y] \leq \inf_{y\geq \sup B} \mathbb{E}[L|y]$ *for all $N \geq N_0$.*

Assumption 3.6. implies that the neighbourhoodof the q^{th} quantile of the random variable $\mathbb{E}[L|Y]$ is connected to the neighbourhood of the q^{th} quantile of Y.

Theorem 3.7. *Under Assumptions 3.2. (2) and 3.6. we have for $N \geq N_0$*

$$\mathrm{VaR}_q(\mathbb{E}[L|Y]) = \mathbb{E}[L|\mathrm{VaR}_q(Y)]. \tag{26}$$

For a proof see [20, Prop 4]. Said in words, under certain assumptions VaR of a random variable $\mathbb{E}[L|Y]$ is equal to $\mathbb{E}[L|\mathrm{VaR}_q(Y)]$ and

$$\mathbb{E}[L|\mathrm{VaR}_q(Y)] = \sum_{n=1}^{N} \mathrm{EAD}_n \cdot \mathrm{LGD}_n \cdot \mathrm{PD}_n(\mathrm{VaR}_q(Y)), \tag{27}$$

where $\mathrm{PD}_n(\cdot)$ is defined as in (18). This central result is the approximation for $\mathrm{VaR}_q(L)$ under single-factor ((13) with $Y_n \equiv Y$ for all n) model.

It is in general not clear how to determine N_0 from Assumption 3.6. and Theorem 3.7. Thus an open question is how big (in the number of loans) a portfolio must be in order for (27) (which is always true in the ASRF case) to be actually a "good" estimate of $\mathrm{VaR}_q(L)$ under the single-factor model, Assumption 3.2. point 2.

In the beginning of this subsection and in subsection 2.3. we outlined the drawbacks of ASRF model and the importance of considering concentration risks, Definition 2.7., respectively. In the following two sections [1] and [3] methodologies are presented, respectively, which try to quantify concentration risk and thus improve the single-factor model VaR given by (27).

4 Cespedes et al. methodology

We begin with an introduction of a semi-analytic model [1] for calculating multi-factor credit risk and measuring sector concentration. Their focus risk measure is the economic capital (EC) defined as

$$\mathrm{EC} = \mathrm{VaR}_\alpha - \mathbb{E}[L]. \tag{28}$$

EC is used in case the expected loss $\mathbb{E}[L]$ is already incorporated in the banking service price. By Assumption 3.2. ASRF model neglects diversification effects in terms of simplifying the dependence structure both on borrower and sector level. Authors provide an extension of the ASRF model to a general multi-factor setting which can recognize diversification effects. They derive an adjustment to the single risk factor model in form of a scaling factor to the economic capital required by the ASRF model. This so-called capital diversification factor (DF) is a function depending on sector size and sector correlations of a particular portfolio. Loan homogeneity is reflected by an index similar to the Herfindahl-Hirschmann-Index (HHI).

Note 4.1. HHI is a market concentration index. It equals the sum of squares of the relative firm size with respect to the total considered market size. In portfolio theory HHI reflects the effective number of loans, i.e., it reaches $1/N$ if a portfolio is composed out of N equal size loans and increases up to one in case there are several or in the extreme case one dominant size loan.

The diversification factor is estimated numerically using Monte-Carlo simulations.

4.1 Basic setup

Our starting point is the general credit loss L, defined by (1), and an asset return model similar to (13)

$$X_n = r_{k(n)} Y_{k(n)} + \sqrt{1 - r_{k(n)}^2}\, \xi_n \quad \text{for all } n. \tag{29}$$

Thus borrower n and m share the same factor loading $r_{k(n)}$ and systematic risk factor $Y_{k(n)}$ if they represent the same sector, i.e., $k(n) = k(m)$. To simplify notation, we write for all k

$$Y_k = Y_k(n), \forall n \in \text{Sector k}.$$

Similarly set the notation $r_k = r_{k(n)}$. For the Y_k dynamics the single macro-factor models (14) and (15) are chosen.

We refer to the β or β_k, (14) and (15), as to the inter- (or cross-) sector correlations and to the r_k as to the intra-sector correlations.

Remark 4.1. Having (29) and (15) the correlation between borrowers n and m (belonging to sectors l and k respectively) asset returns are given by

$$\mathrm{Corr}(X_n, X_m) = \begin{cases} r_k^2 & \text{if } k = l, \\ r_l r_k \beta_l \beta_k & \text{if } k \neq l. \end{cases} \tag{30}$$

Observe that on a sector level point 2 of Assumption 3.2. holds. This allows to approximate subportfolio's, consisting of sector k loans, VaR via an ASRF model. More precisely, at first sector level L is approximated by conditional expectation $\mathbb{E}[L|Y]$ using Theorem 3.5. and then VaR of $\mathbb{E}[L|Y]$ is calculated with Theorem 3.7. Denote sector's k q^{th} percentile VaR by $\text{VaR}_{k,q}$ and equivalently to (27) get

$$\text{VaR}_{k,q} = \sum_{j \in \text{Sector } k} \text{EAD}_j \cdot \text{LGD}_j \cdot \Phi^{-1}\left(\frac{\Phi^{-1}(\text{PD}_j) + r_k \Phi^{-1}(q)}{\sqrt{1 - r_k^2}}\right). \tag{31}$$

Note that this is a strict equality only if the number of loans in sector k is greater than N_0 from Theorem 3.7. Yet we neglect this fact and always write an equality. Sector level economic capital, (28), is then

$$\text{EC}_k = \sum_{j \in \text{Sector } k} \text{EAD}_j \text{LGD}_j \left[\Phi^{-1}\left(\frac{\Phi^{-1}(\text{PD}_j) + r_k \Phi^{-1}(q)}{\sqrt{1 - r_k^2}}\right) - (PD)_j\right]. \tag{32}$$

Assumption 4.2. *Assume perfect correlation between all the sectors, i.e.,* $\beta = 1$ *or* $\beta_k = 1$ *for all* k*, which is then equivalent to the ASRF model systematic risk setup.*

Then an approximation (27) of Theorem 3.7. can be applied to the whole portfolio. The overall capital is then equal to the sum of the stand-alone capital of all sectors

$$\text{EC}^{1\text{f}} = \sum_k \text{EC}_k, \tag{33}$$

where 1f stands for one-factor, due to Assumption 4.1. Equivalently, (by adding back the expected loss) portfolio VaR

$$\text{VaR}^{1\text{f}} = \sum_k \text{VaR}_k. \tag{34}$$

Remark 4.3. Clearly Assumption 4.1. leads to a significant simplification of the underlying loan dependence structure. Nevertheless, VaR^{1f} was at the core of Basel II, [4], regulatory framework for the credit risk capital calculation. In order to compensate for the ASRF model assumption, BCBS introduced additional rules for the parameter choice, e.g., the intra-sector correlation parameters r_k, which stem from calibration procedures with respect to "different" real portfolios.

4.2 The capital diversification factor

We come to the core idea of the methodology [1].

Definition (DF) 4.4. The capital diversification factor DF is defined as the ratio of the economic capital computed using the multi-factor setting and the one-factor capital

$$\mathrm{DF} = \frac{\mathrm{EC}^{\mathrm{mf}}}{\mathrm{EC}^{\mathrm{1f}}}, \tag{35}$$

with $0 \le DF \le 1$. Depending on the problem setup, EC can be replaced by VaR, leading to an analogous DF definition.

Once DF is at hand, the multi-factor VaR or $\mathrm{EC}^{\mathrm{mf}}$ of some portfolio of interest P^* can be calculated as

$$\mathrm{EC}^{\mathrm{mf}}(P^*) = DF(P^*) \cdot \mathrm{EC}^{\mathrm{1f}}(P^*), \tag{36}$$

where $\mathrm{EC}^{\mathrm{1f}}(P^*)$ is calculated using (27) and (6). [1] suggests to estimate DF via a large number of simulated portfolios, for which $\mathrm{EC}^{\mathrm{mf}}$ is approximated via Monte-Carlo and $\mathrm{EC}^{\mathrm{1f}}$ is calculated as before, and by applying linear or nonlinear regression techniques on the gathered data. Thus an equality in (36) will actually be an approximation. In order to parametrize DF, some measures need to be considered, which reflect portfolio composition and the underlying dependence structure. [1] chooses to parametrize DF based upon two following diversification sources:

1. the average inter-sector correlation $\bar{\beta}$,
2. relative sector contribution to $\mathrm{EC}^{\mathrm{1f}}$, captured by the capital diversification index CDI

$$\mathrm{CDI} = \frac{\sum_k \mathrm{EC}_k}{(\mathrm{EC}^{\mathrm{1f}})^2}, \tag{37}$$

where EC_k, $\mathrm{EC}^{\mathrm{1f}}$ can be replaced by VaR_k, (31), $\mathrm{VaR}^{\mathrm{1f}}$, (34), respectively, if the problem setup considers VaR as a risk measure.

One can interpret the ratio 1/CDI as the effective number of sectors in the portfolio. It is a modification of a Herfindahl-Hirschman index, [7], based on the economic capital required in each sector by a single-factor model. It does not capture individual estimates of intrasector and intersector correlations, [11]. Choice of this explanatory variable is also motivated by the fact that, if credit losses were normally distributed (this is not the general case due to D_n, (2)), an equation

$$\mathrm{EC}^{\mathrm{mf}} = \sqrt{(1-\gamma)\mathrm{CDI} + \gamma} \cdot \mathrm{EC}^{\mathrm{1f}} \tag{38}$$

would hold, where γ denotes the single correlation parameter of credit losses (and not the asset correlations). This motivates for the following setup of (36)

$$\mathrm{EC}^{\mathrm{mf}}(\mathrm{CDI}, \bar{\beta}) \approx \mathrm{DF}(\mathrm{CDI}, \bar{\beta}) \cdot \mathrm{EC}^{\mathrm{1f}}, \tag{39}$$

where we of course do not expect a precise statement. Thus the problem reduces to finding an expression for DF in terms of CDI and $\bar{\beta}$. If this is achieved, we are in position to approximate the multi-factor credit risk capital (defined in terms of economic capital) for any portfolio P^* via

$$\mathrm{EC}^{\mathrm{mf}}(P^*) \approx \mathrm{DF}(\mathrm{CDI}(P^*), \bar{\beta}(P^*)) \cdot \mathrm{EC}^{1\mathrm{f}}(P^*). \tag{40}$$

Remark 4.5. EC can be replaced by VaR in (39), as mentioned in Definition 4.4. This implies a different DF in general.

Remark 4.6. ASRF framework can also be used as a crude approximation of a loss model incorporating (14), (15) or even (16) and having r_n defined on a borrower instead of a sector level. It is straightforward to perform Monte-Carlo simulations in any of the previously mentioned cases, hence an approximation of $\mathrm{EC}^{\mathrm{mf}}$ is not an issue. Thus we can provide an approximation of type (39) in any systematic risk factor model case. Depending on the systematic risk model, the average inter-sector correlation $\bar{\beta}$ is defined as:

- the inter-sector correlation β from (14) in case this asset returns model is used,
- in case of (15) set
$$\bar{\beta} = \sum_k \frac{\mathrm{EC}_k}{\mathrm{EC}^{1\mathrm{f}}} \cdot \beta_k, \tag{41}$$
- and, if the systematic risk is given by (16), define
$$\bar{\beta} = \frac{\sum_{k=1}^K \sum_{l \neq k} \theta_{kl} \cdot \mathrm{EC}_k \cdot \mathrm{EC}_l}{\sum_{k=1}^K \sum_{l \neq k} \mathrm{EC}_k \cdot \mathrm{EC}_l}, \tag{42}$$
where $\theta_{kl} \in \Theta$ for a matrix $\Theta \in \mathbb{R}^{K \times K}$. An entry θ_{kl} denotes the correlation between sectors k and l (for instance, correlation of two indices, describing sector k and l performance, respectively).

Similarly to the CDI, (37), one can interchange EC and VaR in equations (41) and (42), if the problem setup uses VaR as the risk measure.

4.3 Parametrization of DF

In this subsection the parametrize procedure of DF, (39), is presented. We will see that the parametrization procedure relies on several assumptions concerning different model parameters. The choice for these assumptions are discussed later in subsection 4.5 and section 6. Let us focus in this case on $\mathrm{VaR}_q(L)$ calculation. Thus we use VaR_k instead of EC_k in the definition of CDI, (37).

The approach is based on Monte-Carlo simulations within the following procedure:

1. Choose the model for asset returns X_n and systematic risk factors Y_k. Fix the asset return and the underlying systematic risk factor model, e.g., (29) with (14).

2. Fix the number of loans and sectors $N, K \in \mathbb{N}$, respectively. Assume that loans are homogeneously distributed across sectors, i.e., $\approx N/K$ loans per sector.

3. Sample β from some assumed distribution, e.g., $\beta \sim \mathcal{U}(0.5, 0.8)$.

4. Simulate independently EADs, PDs, LGDs and r_ks from some assumed distributions. Let us denote by P^* the portfolio that has been created out of steps 2., 3. and 4.

5. Sample "many" times portfolio P^* loss L_{P^*}, (1), via Monte-Carlo simulations:

   (a) sample $Z, \eta_k \sim \mathcal{N}(0, 1)$ for every sector k,

   (b) calculate the risk factor value $Y_{k(n)}$,

   (c) sample ξ_n for every loan n, calculate X_n, (29), for all n,

   (d) finally calculate L, (1).

6. Estimate $\mathrm{VaR}_q^{\mathrm{mf}}(L_{P^*})$ based on the obtained sample data for L_{P^*} .

7. Calculate $\mathrm{VaR}_q^{\mathrm{1f}}(L_{P^*})$ as in (34).

8. Calculate $\mathrm{CDI}(P^*)$, (37), and $\bar{\beta}(P^*)$ according to the chosen model (see Remarks 4.5. and 4.6.).

9. Set
$$\mathrm{DF}(\mathrm{CDI}(P^*), \bar{\beta}(P^*)) = \frac{\mathrm{VaR}_q^{\mathrm{mf}}(L_{P^*})}{\mathrm{VaR}_q^{\mathrm{1f}}(L_{P^*})}. \tag{43}$$

10. Save DF, CDI and $\bar{\beta}$ values for P^* and go to step 3. to construct new portfolio and re-do steps 4.-10. Once DF, CDI and $\bar{\beta}$ values are obtained for "reasonably" many portfolios, we can proceed with DF parametrization.

11. The diversification factor DF is a function of $\bar{\beta}$ and CDI. Thus DF defines a 3D surface over the $\bar{\beta} \times \mathrm{CDI}$ plane. DF parametrization can be obtained, for instance, using non-linear regression, where one has to estimate parameters a_{ij} for $i, j = 1, 2, \ldots, C$ based on the dataset obtained in step 10, assuming that DF follows the rule
$$\mathrm{DF} = a_0 + \sum_{i,j \geq 1} a_{ij} (1 - \mathrm{CDI})^i (1 - \bar{\beta})^j. \tag{44}$$

    Let $C = 2$ be fixed for all further DF models. After DF parametrization is obtained, one typically chooses $(\bar{\beta}, \mathrm{CDI}) \in [0, 1]^2$ to plot the DF surface. This is due to the construction of CDI, implying $0 < \mathrm{CDI} \leq 1$, and since firms usually show positive correlatedness in their performance reaction on macroeconomical changes or on bankruptcy of binding companies, motivating for nonnegative $\bar{\beta}$. Of course, the $\bar{\beta}$ range for DF surface plot can be extended if steps 3 & 10 argue in favour of that, yet this is not the case of examples contained in this thesis.

4.4 Comments on DF parametrization

DF parametrization relies on time-consuming Monte-Carlo simulations (recall that we perform step 5 for every artificially constructed portfolio P^*). Since we want to achieve high approximation precision for $\mathrm{EC}^{\mathrm{mf}}$, we would need our artificial portfolios to densely reflect a setup region (in the sense of number of loans and sectors, specifications of EADs, PDs etc.) of prospective future portfolios. Thus, it is important to have *a priori* a good understanding of how to sample (from which distributions in order to reflect potential portfolio characteristics) in steps 3 & 4 and what are/will be the typical number of loans and sectors, step 2.

See Remark 4.8. for the discussion concerning the homogeneous loan distribution assumption in step 2.

A popular statistical estimator of $\mathrm{VaR}_q(L)$, recall step 6, is $L_{\lceil qI \rceil}$ obtained by increasingly ordering the samples $(L_i)_{i\in\{1,\ldots,I\}}$, obtained in 5. This estimator is used in numerical experiments, presented in section 4.6.

The assumed probability distributions for parameter sampling in steps 3 & 4 may be either continuous or discrete. Dependence across some or even all parameters can be imposed in their sampling. If needed, some parameters may be held constant or defined as a function of the others. See some related remarks in the discussion of subsection 4.5.

Depending on the chosen model (step 1) and a specific portfolio construction (steps 2-4), the corresponding loss variable L can possess a heavy tailed density function. This can imply poor VaR_q estimations for q close to 1 unless "considerably large" number of samples were computed (step 5).

One can choose a parametrization model different to (44). Sometimes a constraint $a_0 = 1$ (together with model (44)) is set in order to stress the upper bound $\mathrm{DF} \leq 1$. Yet the parametrization model (44) has shown to fit nicely (in the sense of statistical tests) sampled data clouds (step 10) in different numerical experiments. See subsection 4.6 for practical examples.

Finally, after doing the procedure of DF parametrization from subsection 4.3, in which the risk parameter characteristics were chosen such to reflect some specific needs, the time consuming calculation has to be carried out once. The obtained results and DF are stored and can be used as long as the future portfolio compositions are not in "serious" contrast to the simulated portfolios, used for DF parametrization.

4.5 Critique and extensions in the literature

The main benefit of [1] is the fast calculation and simple expressions once DF is calibrated. [1] also suggests simple marginal capital contribution evaluation technique, which is an additional risk management instrument that can contribute to a better portfolio credit risk monitoring.

Several immediate drawbacks need to be mentioned. First of all it is the method's reliance on the ASRF model, which fully neglects dependence structure across asset returns. The explanatory variables CDI and $\bar{\beta}$ capture to some extent the concentration risk and the average cross sector correlation. This cannot recognize name concentration risks, i.e., when different loans assigned to one or several sectors exploit higher downward movement correlations than other loans of the same sector. This can happen if, for instance, different companies are units of some greater holding. Furthermore, the model doesn't allow for borrower specific asset correlations, see Remark 4.1. Yet a crude estimation, for instance, based on weighted average $\hat{r_k}$ of r_n corresponding to loans from sector k can be constructed, as discussed in Remark 4.6.

Another drawback is the simplifying assumption of the average cross-sector correlation $\bar{\beta}$. Whereas it is used as a portfolio characterizing measure and thus one would want it to distinguish different portfolios, this is not the case in general. For instance, a portfolio, which is highly concentrated towards a sector with a high correlation with other sectors, and another portfolio, which is equally high concentrated, but towards a sector, which is only weakly correlated with other sectors, can possess the same average cross-sector correlation. However, the concentration risk levels in such portfolios can be considerably different. This is also noted in [3].

DF parametrization, subsection 4.3, requires Monte-Carlo simulations. In practice one may work with credit portfolios of size greater than 4000 loans. The nature of each portfolio can be different. Each portfolio may have individual regional or industrial sector based systematic risk factors, different relative exposure sizes and dependence relations across risk parameters as EADs, PDs, r_k, etc. Thus to obtain good DF quality, large amount of artificially simulated portfolios for DF parametrization need to be considered. This may lead to, e.g., days or even weeks of compilation procedures in Matlab on a standard PC.

[1] methodology has been also criticized for the relation $\mathrm{DF} \leq 1$, definition 4.4. The reason is the following. ASRF model is used for the credit risk capital requirement calculation in the Basel II, [4], regulatory framework. Yet to compensate for many assumptions of the ASRF model, a calibration with respect to some real life benchmark portfolios was performed, making $\mathrm{VaR}^{\mathrm{1f}}$ a good approximation of $\mathrm{VaR}^{\mathrm{mf}}$ for the benchmark portfolios. Thus, as a result of calibration, some relations for risk parameters as factor loadings r_n, PDs, LGDs, etc., were implied. For example, calibration implied relations between PDs and r_n, which are prescribed in [4] regulations but at the same time are doubted by many empirical studies, as noted in [13]. Apart from that, assume one incorporates these relations in steps 2-4, subsection 4.3,, when simulating artificial portfolios for DF parametrization. Let us call $\bar{\mathrm{DF}}$ the resulting DF parametrization for this case. Then $\bar{\mathrm{DF}} \leq 1$ does not hold in general. $\bar{\mathrm{DF}} \approx 1$ for the portfolios used in calibration. This implies that for any more diversified portfolio $\bar{\mathrm{DF}} \leq 1$, whereas for a portfolio with higher concentration risks and less homogeneous loan exposure sizes one obtains $\bar{\mathrm{DF}} \geq 1$. Also authors of [1] recognize this possible drawback and suggest that a case dependent rescaling factor for DF can be introduced to account for this issue. See, for instance, [12] for relevant results.

4.6 Parameter sensitivity test and discussion

In this subsection examples for the DF parametrization procedure from subsection 4.3. are presented. We give plots to observe the impact of changes in the underlying parameters, that determine portfolio. An additional task is to induce a better intuitive understanding of Monte-Carlo sampling results.

Assume we are interested in the 99.9% percentile level $\mathrm{VaR}^{\mathrm{mf}}$. We follow the prescription of subsection 4.3 to obtain the corresponding DF.

Example 1. To start with a relatively fast numerical experiment, let us simulate 280 portfolios under the following conditions (see Remark 4.7. on discussion concerning the choice of parameter sampling distributions).

- Choose the asset returns model (29) together with the single macro factor systematic risk factor model (14).
- Set the total number of loans to $N = 1000$ and let the number of loans per sector be equal to K/N (with an appropriate rounding) for $K = 1, 2, 3$.
- Sample inter-sector correlation parameter $\beta \sim \mathcal{U}(0, 1)$.
- Let $\mathrm{PD}_k \sim \mathcal{U}(0.01, 0.075)$ for every sector k and set $\mathrm{PD}_n = \mathrm{PD}_k$ for every loan n from sector k. Thus we assume homogeneous default probabilities across all loans from a particular sector. Although real life portfolios in general do not fulfil this assumption, it is partly motivated by [3], where the methodology assumes sector level aggregation of loans, leading to, for instance, sector level PDs. See subsection 5.2.
- Similarly to the above definition of PD_k let $\mathrm{LGD}_k \sim \mathcal{U}(0.5, 1)$ and $\mathrm{EAD}_k \sim \mathcal{U}(0, 1)$ independently for every sector k.
- Sample the intra-sector correlation parameters $r_k(n)$ uniformly from the interval $(0.3, 0.6)$.

For each constructed portfolio P^* calculate $100'000$ Monte-Carlo samples of L_{P^*} to estimate $\mathrm{VaR}_q^{\mathrm{mf}}(L_{P^*})$.

Let us comment on the first results. In figure 2 a 2D plot is presented, where each point corresponds to one of 280 artificially constructed portfolios, showing each portfolio $\bar{\beta}$ and CDI. Comments on figure 2:

- The more sectors there are, the "easier" it is to achieve higher diversification level, subsequently lower CDI.
- Having one dominant sector among several can still lead to high CDI and little diversification effect. Red dots cover wide range of CDI, from 0.5, which is the lower bound for $K = 2$, and almost up to 1. For instance, red portfolios with CDI values of around 0.9 possess one dominant sector in terms of loan exposure sizes ($1/\mathrm{CDI} \approx 1.1$).

In figure 3 we present the full result picture stemming from steps 1-10 from subsection 4.3. Every dot corresponds to a portfolio with its CDI, $\bar{\beta}$ and DF,

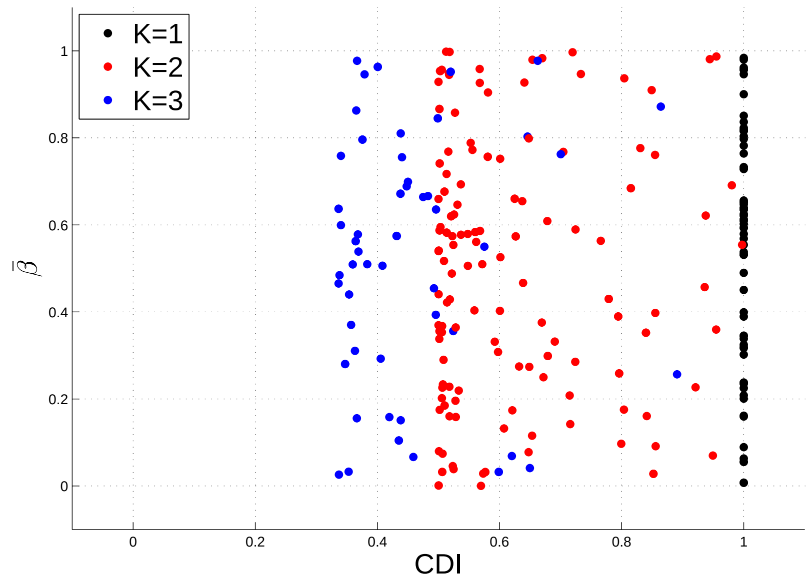

Figure 2: Example 1. Coloured dots are 280 portfolios, each 1000 loans. Loans of each portfolio are distributed across K many sectors. Each portfolio is described by two diversity capturing measures: capital diversification index (CDI, (37) with VaR instead of EC) and the average cross-sector correlation $\bar{\beta}$ (equal to β from (14) as by remark 4.6). Plot visualizes results of step 8, subsection 4.3. A portfolio composed of loans from more sectors is more likely to show better diversity characteristics, i.e., lower CDI and $\bar{\beta}$ values.

(43). Dot color shows number of sectors in the portfolio (see color legend). This data cloud can be used to perform regression analysis as in step 11, subsection 4.3. One can see that portfolios consisting of one sector loans (black dots) show no diversification effects (DF ≈ 1). As expected, DF is increasing in $\bar{\beta}$ and CDI.

Remark 4.7. The choice of the sampled parameter distribution (hence sampling interval) is usually based on historical data. For instance, a common case is when the probabilities of default lie in the range of 1% to 7.5%. Sometimes risk managers use rating agency data and assign to each rating level some default probability. If one has a strong belief in some characteristics of prospective real life portfolios, these assumptions can be taken into account and incorporated into the sampling intervals of different risk parameters, subsection 4.3, steps 2-4, e.g. shifting bounds of uniform distribution. Note that in general correlation of two different asset returns, (13), can be negative, for instance, prices for bonds issued by a gold mining company may rise during financial crisis. Yet the philosophy of splitting loans into sectors suggests to combine highly correlated loans

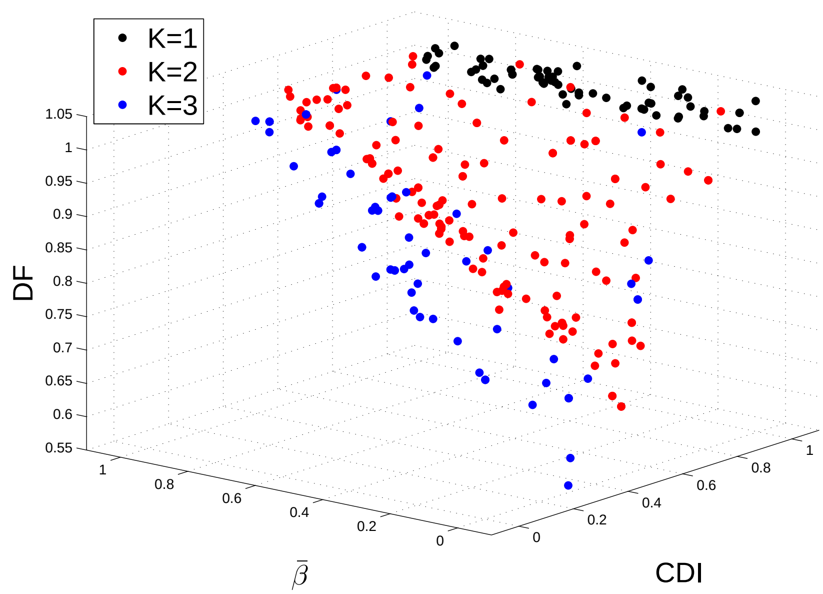

Figure 3: Example 1. Diversification factor (DF, (43)) of each portfolio from figure 2. DF of a portfolio is the ratio between simplified ASRF model VaR value VaR^{1f}, (34), and the full multi-factor model VaR value $\mathrm{VaR}^{\mathrm{mf}}$, estimated using Monte-Carlo approach. Diversity capturing measures CDI and $\bar{\beta}$ as in figure 2. Multi-factor VaR is lower than VaR^{1f} for portfolios, that provide some diversity (CDI < 1 and $|\bar{\beta}| < 1$). DF decreases with decreasing CDI and $\bar{\beta}$ values.

in one sector and to have presumedly low correlation for loans from different sectors. Due to this we usually sample r_k with positive values.

Remark 4.8. The assumption of homogeneous distribution of loans across sectors (thus having $\approx N/K$ loans per sector) is not a restrictive assumption. If one works with portfolios, in which several dominant loans and many small loans are likely to occur, i.e. with high CDI, the sampling procedure for EADs can be changed to, for instance,

$$\mathrm{EAD}_n = e^{a_n}, \quad \text{where } a_n \sim \mathcal{U}(0, 10). \tag{45}$$

Example 2. Left hand side plot in figure 4 presents similar result to figure 2, but with risk parameters PD, EAD and LGD simulated on a loan level, i.e., individually for every loan. As one can observe, portfolios are concentrated more closely to the their CDI lower bounds, 1/3, 1/2 and 1 for $K = 1, 2, 3$, respectively. The reason for that is the high loan number, which results in relatively homogeneous total exposure size of each sector loans. This is important to consider, since with this DF parametrization setup we certainly obtain gaps between simulated portfolios of different total number of sectors. For instance, VaR^{mf} estimation of a 3 sector portfolio, which contains one dominant sector and has CDI of ≈ 0.8, can be very poor if one uses DF based on figure 4 simulations.

The right hand side plot in figure 4 is based on the same framework as the left hand side plot, but with a different choice of EADs, (45). (45) certainly results in relatively wider exposure size spread across the loans. We do observe several blue points with CDI > 0.4 and more red points around CDI ≈ 0.6, which is not the case in the left plot of figure 4, but it still would need dozens of portfolio simulations to cover $\bar{\beta}$-CDI surface more homogeneously.

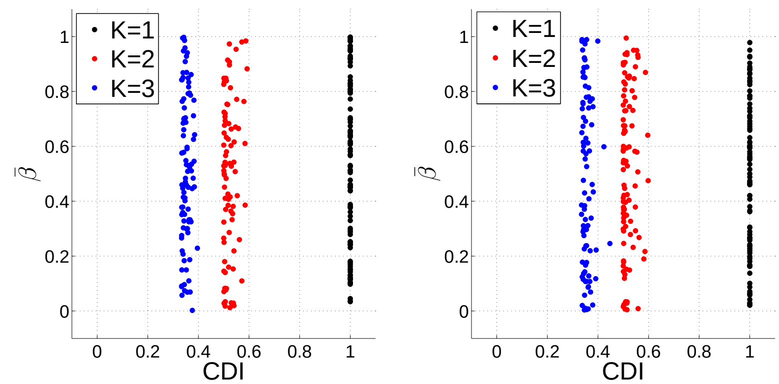

Figure 4: Example 2. Coloured dots are 280 portfolios, each 1000 loans. Loans of each portfolio are distributed across K many sectors. Each portfolio is described by two diversity capturing measures: capital diversification index (CDI, (37) with VaR instead of EC) and the average cross-sector correlation $\bar{\beta}$ (equal to β from (14) as by remark 4.6). Plot visualizes results of step 8, subsection 4.3. In contrast to Example 1, loan exposures at default (EAD), losses given default (LGD) and probabilities of default (PD) are sampled individually for every loan instead of sector. On the left: $\text{EAD}_n \sim \mathcal{U}(0, 1)$. On the right: EAD_n as in (45). Sampled portfolios are concentrated close to their CDI lower bounds, $\approx 1/K$, even for (45). As a results, poor coverage of the surface for CDI $\in (1/3, 1)$.

Example 3. Figure 4 showed that (45) has little effect in case EADs are sampled on a loan level. The reason is that big exposure size portfolios are likely to appear in every sector. This would induce problems to reasonably cover unit square of CDI $\times\, \bar{\beta}$ plane. One can fix that by sampling EADs using (45) on a sector level, as in Example 1. This increases the likelihood of getting one or several dominant sectors, thus obtaining representatives with wider variety in CDI. Figure 5 confirms this reasoning. The top left plot of figure 5 visualizes first 140 portfolio simulations and additional 140 portfolios are added in each following plot from left to right, top to bottom.

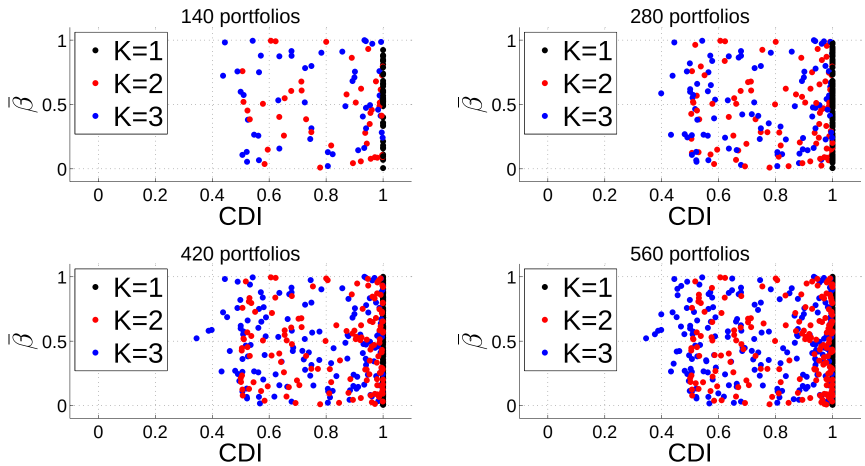

Figure 5: Example 3. Each portfolio (coloured dot) is described by two diversity capturing measures: capital diversification index (CDI, (37) with VaR instead of EC) and the average cross-sector correlation $\bar{\beta}$ (equal to β from (14) as by remark 4.6). Plot visualizes propagation of artificially sampled portfolios in the CDI-$\bar{\beta}$ plane (as the number of sampled portfolios increases) using subsection 4.3 scheme. Exposure sizes (EAD) sampled on a sector level, using (45). Now high CDI value portfolios appear more often even in case $K = 3$. Better surface coverage (for CDI ≥ 0.4) compared to figure 4.

Example 4. Let us choose single macro factor systematic risk model (14) and randomly simulate number of sectors for each portfolio in the range $K \in \{2, 3, \ldots, 12\}$. Let the total number of loans be fixed, $N = 1200$, which are homogeneously distributed across all sectors in particular portfolio setup, i.e., $\approx N/K$ loans per sector. Risk parameters are simulated using the corresponding distributions for β, PD, EAD, LGD and $r_k(n)$ of Example 1 on the sector level. Sector level exposures EAD_k for every sector k are normalized to w_n, (5). $10'000$ portfolios are simulated. In figure 6 $\mathrm{VaR}^{\mathrm{mf}}$ is estimated using $100'000$ Monte-Carlo simulations, whereas in figure 8 $500'000$ Monte-Carlo simulations (thus additional $400'000$ samples of the loss variable L_P, (1), for every portfolio P) are used. Observe that in figures 6 and 8 CDI of the artificial portfolios do not fall below $\approx 1/12$. The reason is the maximum sector $K = 12$, which implies a CDI lower bound of $1/12$ for any related portfolio. Denote by DF_S and DF_B the DF parametrizations stemming from two cases, which are visualized in figures 6 and 8, respectively.

$$
\begin{aligned}
\mathrm{DF}_S = & 1.0083 - 0.92159(1 - \mathrm{CDI}) \cdot (1 - \bar{\beta}) + 0.10614(1 - \mathrm{CDI})^2 \cdot (1 - \bar{\beta}) \\
& + 0.44788(1 - \mathrm{CDI}) \cdot (1 - \bar{\beta})^2 - 0.44431(1 - \mathrm{CDI})^2 \cdot (1 - \bar{\beta})^2 \qquad (46)
\end{aligned}
$$

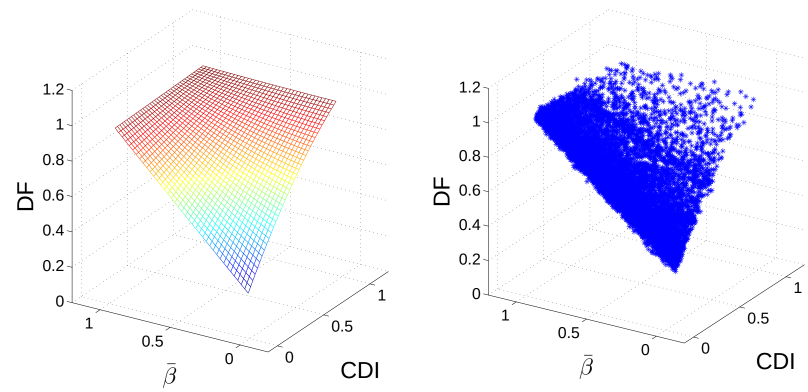

Figure 6: Example 4. Diversification factor DF_S, (46), surface plot obtained using parametrization model (44) based on $10'000$ simulated portfolio data, represented with blue stars on the right. $\mathrm{VaR}^{\mathrm{mf}}$ of each portfolio is estimated using Monte-Carlo approach with $100'000$ corresponding loss variable L, (1), samples. Diversity capturing measures: capital diversification index (CDI, (37) with EC replaced by VaR) and the average cross-sector correlation $\bar{\beta}$ (equal to the sampled inter-sector correlation β, (14), as by remark 4.6). DF increases for higher CDI and $\bar{\beta}$ values.

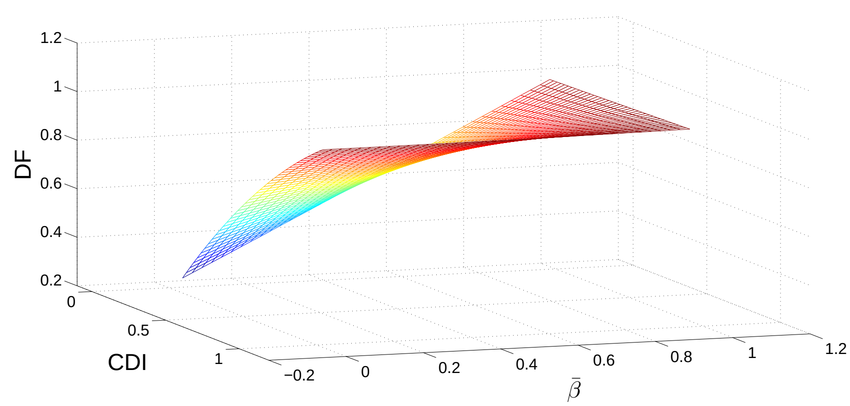

Figure 7: Example 4. Rotated DF_S, (46), surface from figure 6.

DF parametrization (46) shows 0.988 adjusted R^2 statistics and less than $2e{-}09$ p-value for all parameters a_0 and a_{ij} (see step 11., subsection 4.3).

$$\begin{aligned}\mathrm{DF}_B = & 1.0081 - 0.91883(1-\mathrm{CDI})\cdot(1-\bar{\beta}) + 0.10361(1-\mathrm{CDI})^2\cdot(1-\bar{\beta}) \\ & + 0.44502(1-\mathrm{CDI})\cdot(1-\bar{\beta})^2 - 0.44117(1-\mathrm{CDI})^2\cdot(1-\bar{\beta})^2 \qquad (47)\end{aligned}$$

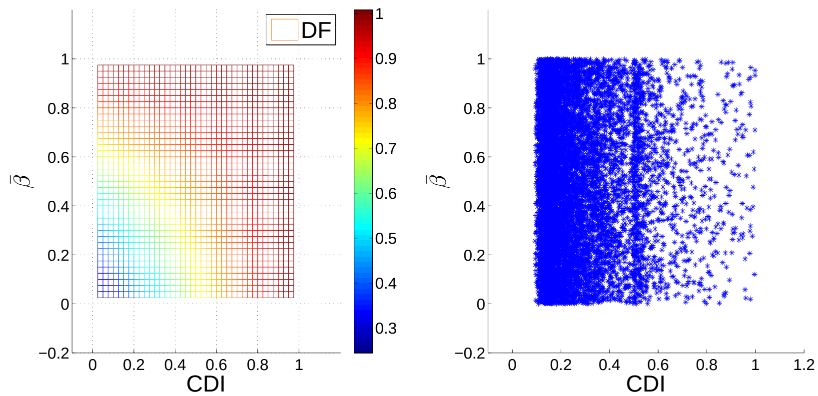

Figure 8: Example 4. Diversification factor DF_B, (47), surface based on parametrization (44) using $10'000$ simulated portfolio data. Each portfolio represented with a blue star and its $\mathrm{VaR}^{\mathrm{mf}}$ is estimated using Monte-Carlo approach with $500'000$ corresponding loss variable L, (1), samples. Definitions of CDI and $\bar{\beta}$ the same as in figure 6. DF_B surface shape is close to figure's 6. To provide a wider overlook, a different angle perspective is offered. Figures 9 and 10 capture the impact of the increased number of Monte-Carlo samples on the DF_B parametrization compared to DF_S, (46).

DF parametrization (47) shows 0.992 adjusted R^2 statistics and less than $5e-12$ p-values for all parameters a_0 and a_{ij} (see step 11., subsection 4.3).

To present the difference of DF_S and DF_B, we construct a 100×100 test point grid in the $\bar{\beta}$-CDI $[0,1]^2$ surface and compare the two DF. Let us study the ratio $\mathrm{DF}_S/\mathrm{DF}_B$, visualized in figure 9. Ratio has mean 1, median 1.001 and standard deviation 1.6141e-04. Maximum and minimum observed ratio values are 1.0002 and 0.9984, respectively. Figure 9 shows that in more than 65% of the cases the ratio is above 1, meaning that DF_B usage results in smaller capital requirements. Still we observe much wider spread of values to the left of 1. In figure 10 we colour blue the grid points with $\mathrm{DF}_S/\mathrm{DF}_B \geq 1$.

One should not get confused from the following observation. From figures 9 and 10 we conclude that for less diversified portfolios, i.e., higher CDI and $\bar{\beta}$ values, DF_B, (47), imply lower capital requirements in terms of VaR. One may find this result counterintuitive, since (35) implies smaller $\mathrm{VaR}^{\mathrm{mf}}$ using $500'000$ samples instead of $100'000$, whereas in contrast to that less diversified portfolios tend to have heavier tail distributions, as shown in section 6, and one may expect $\mathrm{VaR}^{\mathrm{mf}}$ to slightly increase with increasing number of samples. This observation is not justified because, first of all, we do not get an equality (35), as discussed earlier in subsection 4.2. Secondly, as shown in figure 6 and 8, most of the artificially constructed portfolios fall into the white region and around the white-blue boundary of figure 10. This implies that actually the white region of figure 10 is densely presented with artificial portfolios and the blue region of

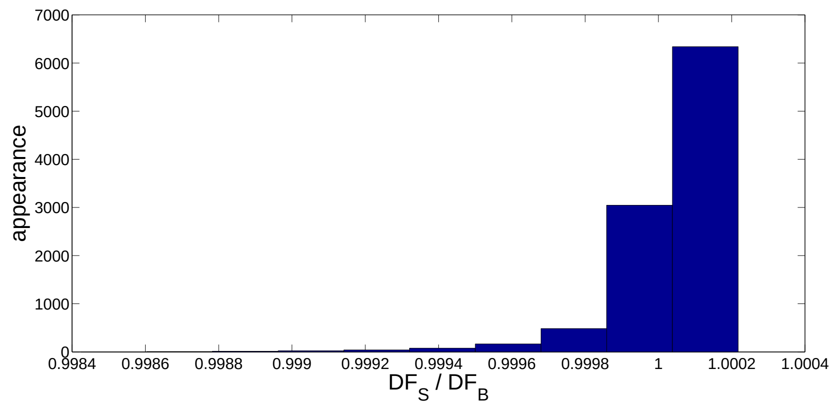

Figure 9: Example 4. Value appearance frequency histogram of the ratio $\mathrm{DF}_S/\mathrm{DF}_B$, based on 100×100 point-grid on the $[0,1]^2$ square in the CDI-$\bar{\beta}$ (step 8, subsection 4.3) plane. Diversification factor parametrizations defined in (46) and (47), respectively.

the same figure is more of a nonlinear regression result and one should try to avoid using DF_B or DF_S for that region portfolios. See Example 5 for related discussion. In particular, $\mathrm{VaR}^{\mathrm{mf}}$ estimations based on larger samples need to be taken as a reference. All in all, this implies that we consider DF_B as a reference for DF_S and conclude that DF_S underestimates the multi-factor VaR for well-diversified portfolios and overestimates it for less-diversified portfolios.

This example motivates to consider improved Monte-Carlo methods, in order to increase the estimation quality and reduce its variance without significantly increasing computational time. For instance, DF_B (47) calculation took approximately 34 hours on the ETH Zürich supercomputer Brutus. See section 7 for variance reduction methods to improve MC performance and a description of the ETH Zürich supercomputing infrastructure.

Example 5. In this example we present the impact of the inter-sector correlation matrix on the DF parametrization procedure. Intuitively it is clear that a good prior estimation of the risk parameter distributions, sector types and the related correlations (inter-sector correlation matrix) of the prospective portfolios is important to achieve better $\mathrm{VaR}^{\mathrm{mf}}$ estimations. Knowing these characteristics allows us to simulate less portfolios, thus reduces computational time without increasing the error. This is due to the fact that the artificially simulated portfolios will be concentrated in the region where one expects to have future portfolios. Mathematically expressed, let the systematic risk factor model be defined as in (16). Let the total number of sectors be $K = 12$ and $\Theta_1, \Theta_2, \Theta_3, \Theta_4 \in \mathbb{R}^{K\times K}$ represent 4 different inter-sector correlation matrices[2].

[2]See Appendix A for explicit Θ_i representations, $i = 1, 2, 3, 4$.

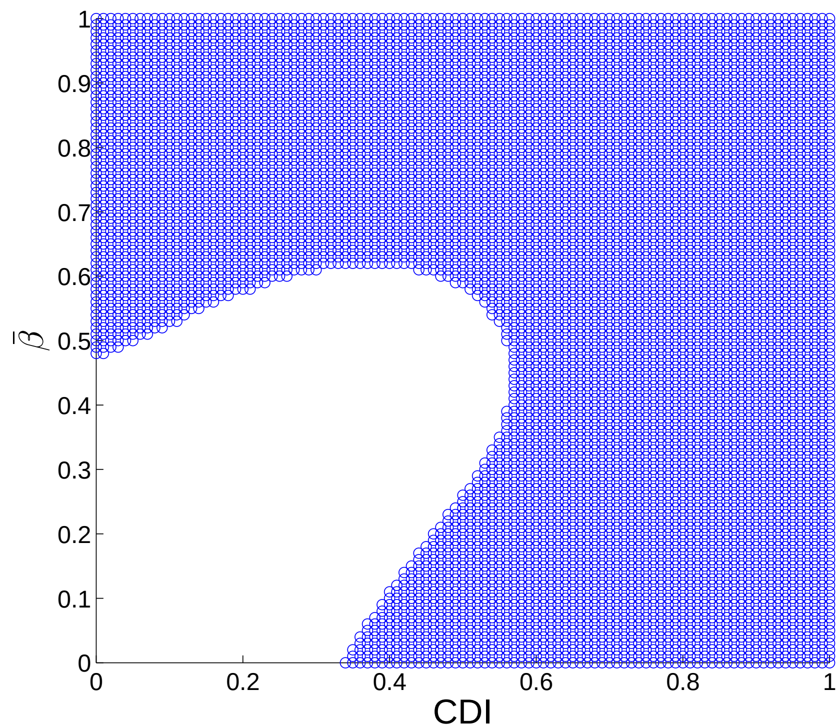

Figure 10: Example 4. A supplement to histogram from figure 9. Blue zone corresponds to the region with $\mathrm{DF}_S/\mathrm{DF}_B \geq 1$. The diversification effect of DF_S is overestimated for well-diversified portfolios and underestimated for less diversified portfolios.

Let us introduce additional names `nCorr` for Θ_1, `lCorr` for Θ_2, `mCorr` for Θ_3 and `hCorr` for Θ_4 to identify the underlying no-, low-, medium- or high relative correlation level across K sectors, respectively. Recall that the parameters α_{ij} in equation (16) stem from the lower triangular Cholesky decomposition of the inter-sector correlation matrix, for $i, j = 1, 2, \ldots, K$. Let the risk parameters be simulated as in Example 1. In figure 11 we visualize the results of steps 3.-10. from subsection 4.3, i.e., artificially constructed portfolios that are afterwards used for DF parametrization.

First comment on figure 11 is that the average cross-sector correlation $\bar{\beta}$, (42), can be also negative. We did not see this in previous examples due to discussion in step 11 of DF parametrization, subsection 4.3., and Remark 4.7. Second comment concerns the yellow point cloud, which is concentrated on $\bar{\beta} = 0$ line. This is clear because nCorr is an identity matrix, thus there is no linear dependence, i.e. correlation, across sectors. Thirdly, in relatively extreme

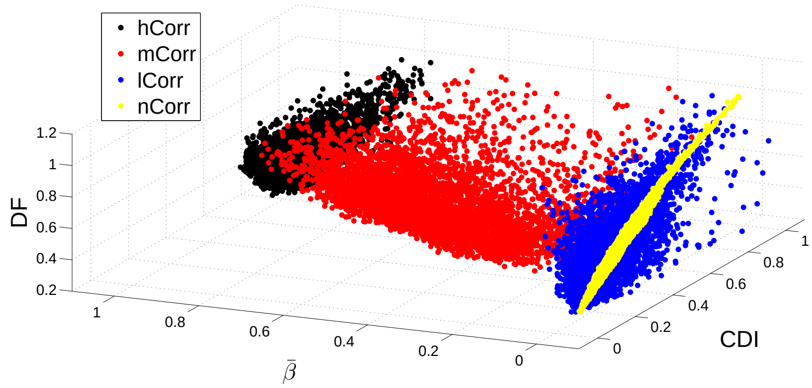

Figure 11: Example 5. Plot visualizes the impact of an inter-sector correlation matrix on the portfolio simulation results. Matrices `xCorr`, $\mathtt{x} \in \{\mathtt{h}, \mathtt{m}, \mathtt{l}, \mathtt{n}\}$ explained in Appendix A. Total number of sectors is $K = 12$ for all portfolios, equal to the dimension of each matrix. Multi-factor adjustment diversification factor DF defined in (43), capital diversification index CDI and the average cross-sector correlation $\bar{\beta}$ as in (37) and (42), respectively (with VaR instead of EC). Observe the impact of the matrix dependent systematic risk factor model (16) on the localization of portfolios in the CDI-$\bar{\beta}$ plane. Important to consider in order to exclude an application of, for instance, DF, (44), based on `hCorr` data cloud on a well-diversified real-life portfolio (with low $\bar{\beta}$ and CDI values).

cases of `lCorr` or `hCorr` we observe that the artificial portfolios are densely concentrated in specific regions, especially in contrast to `mCorr` portfolios, which capture wide spread of $\bar{\beta}$ and CDI combinations. The impact of `hCorr` and `lCorr` portfolios on DF parametrization, step 11. from subsection 4.3, can be observed in figure 12. The difference can be clearly seen without additional calculations. As a result, clearly a high precision in multi-factor VaR estimation via `hCorr` DF will be achieved for portfolios, which fall in `hCorr` artificially sampled portfolio region. Whereas, for instance, it would be unsatisfactory to estimate $\mathrm{VaR}^{\mathrm{mf}}$ for a portfolio with $\bar{\beta} = 0.7$ using `lCorr` DF parametrization, because we had no artificial portfolios there to impact the regression analysis and hence the DF.

Example 6. In this example we present a DF parametrization based on $50'000$ portfolio sample. The resulting DF will be considered as a reference result and used later in the thesis for comparison purposes. We consider it as a reference result in the sense of, firstly, having greatest number of sampled portfolios (used for DF parametrization) among all previous examples and, secondly, we do not *a priori* specify portfolio sampling scheme to some strict contexture, thus our aim is to represent the whole unit square $[0, 1]^2$ in the CDI $\times\, \bar{\beta}$ plane. We choose $\mathrm{EC}^{\mathrm{mf}}$, (28), with $q = 99.9\%$ as a risk measure for this example, thus in contrast to previous examples, where DF was a ratio $\mathrm{VaR}^{\mathrm{mf}}/\mathrm{VaR}^{\mathrm{1f}}$.

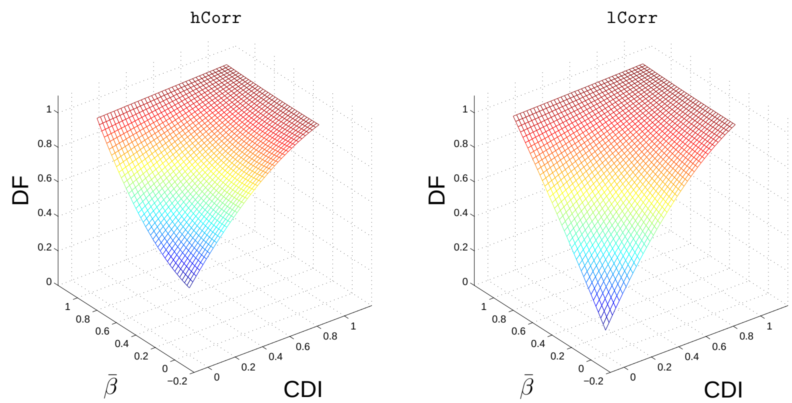

Figure 12: Example 5. Multi-factor adjustment diversification factor (DF) surface plot, (44), based on artificial portfolios stemming from the asset returns model (13) with systematic risk factors (16) defined by inter-sector correlation matrices `hCorr` and `lCorr`. Each case data clouds used for DF parametrization (step 11, subsection 4.3) are visualized in figure 11. Clearly surface shapes are different, thus it is important to know whether the artificially sampled data (used for DF parametrization) and the DF parametrization function model reasonably represents characteristics of a real life portfolio in order to allow its application.

Each portfolio consists of $K = 12$ sectors, 100 loans per sector. Systematic risk model (14) is used. All risk parameters are calculated in the same way as in Example 1. The only difference concerns EAD_k of every sector. As Example 2 has shown, this EAD sampling approach would lead to high concentration of well-diversified portfolios close to 1/12. To better capture wide ranges of CDI, for each portfolio a random number γ is sampled from the multinomial distribution with range $\{0, 1, \ldots, 11\}$ with equal occurrence probabilities for every outcome. Then first γ sector exposure sizes are multiplied by $\Gamma \sim \mathcal{U}(1, 100)$. Thus we impose relative size dominance to first γ sectors. Finally, $\mathrm{VaR}_q^{\mathrm{mf}}$ is estimated using crude MC with $250'000$ samples. See figure 13 for the resulting plot. Resulting parametrization function is

$$\begin{aligned} \mathrm{DF} =& 1.0082 - -0.74533(1 - \mathrm{CDI}) \cdot (1 - \bar{\beta}) - 0.26594(1 - \mathrm{CDI})^2 \cdot (1 - \bar{\beta}) \\ &+ 0.39918(1 - \mathrm{CDI}) \cdot (1 - \bar{\beta})^2 - 0.33344(1 - \mathrm{CDI})^2 \cdot (1 - \bar{\beta})^2 \qquad (48) \end{aligned}$$

and has adjusted R^2 statistics level of 0.992.

4.7 Summary

In this section a scaling factor DF was introduced to correct for the multi-factor VaR of a portfolio, given its ASRF approximation. We began section 4 with the

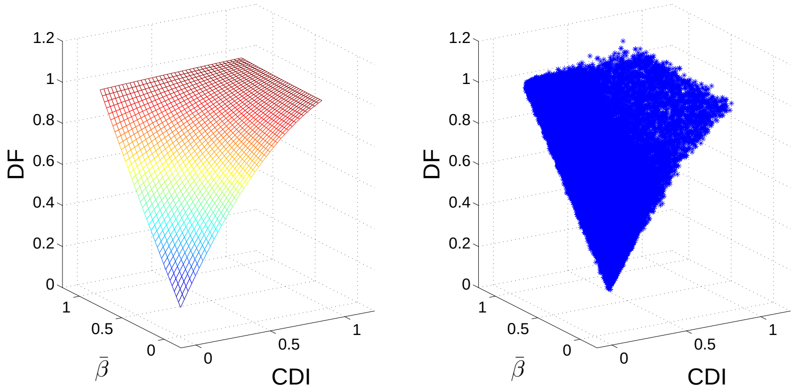

Figure 13: Example 6. Resulting diversification factor (DF) parametrization surface (48), based on $50'000$ simulated portfolios. Blue dots are the sampled portfolios. Each portfolio is described by two diversity capturing measures: capital diversification index (CDI, (37) with VaR instead of EC) and the average cross-sector correlation $\bar{\beta}$ (equal to β from (14) as by remark 4.6). This simulation has the best unit square surface portfolio coverage in the CDI-$\bar{\beta}$ plane (compared to previous simulations), hence the related DF parametrization, (48), will be used for further comparison studies.

description of DF and theoretical motivation. The final result is the presented DF parametrization scheme in subsection 4.3. Due to many assumptions concerning portfolio construction and risk parameter sampling procedures, steps 1-4, we provided different examples in section 4.6. This influences the distribution of artificial portfolios in the CDI $\times$ $\bar{\beta}$ plane, which is of particular interest for the user. In its turn, artificial portfolios influence the result of a non-linear regression, when finally parametrizing DF as a function of CDI and $\bar{\beta}$. One would like the real life portfolio to land in the region, which was represented reasonably dense, thus it is important to know how to sample.

Example 1 provided quick results and developed first impression of the procedure. Apart from number of sectors and loans (often larger in practice), other risk parameter sampling distributions are close to ones, that are often used in practice.

We discuss the question of efficient portfolio sampling in the sense of covering wide ranges of CDI and $\bar{\beta}$ in Examples 2 and 3. The important take-away of this example is that under loan level EAD sampling it is difficult to obtain portfolios, which land far from their CDI lower bound, thus show dominance in one or several sectors.

VaR^{mf} relies on Monte-Carlo estimation, which may show high variance. Apart from that, Monte-Carlo estimations are very time consuming, implying a trade-off between precision and computational time. We study the impact of increased number of MC sample on DF in Example 4. We noticed that in our setup less-diversified portfolio VaR is overestimated by DF_S, (46).

Main point of Example 5 is the change of systematic risk model to (16). This, in its turn, results in a heavy dependence on the sector definitions and related correlation matrices Θ.

For the later comparison purposes we constructed DF, (48), based on biggest number of simulations that we have performed in context of this thesis under [1] methodology. The approach and results are described in Example 6.

5 Düllmann et al. methodology

In this section we review a different analytical approach for estimating credit risk capital defined in terms of VaR or ES of the credit portfolio loss variable L, (1). This, of course, includes the EC case, since $\mathbb{E}[L]$ can be computed analytically. Similarly to Cespedes et al,[1], the goal is to construct an estimator which also recognizes the concentration risks. The approach is also based on the ASRF model and introduces additional granularity adjustment terms within the multi-factor Merton framework. The methodology is based on the 2$^{\text{nd}}$ order Taylor expansion of $\mathrm{VaR}(L)$ around the limiting portfolio loss variable from the ASRF model. It was first introduced by Pykhtin, [8], and then partly simplified and revisited by Düllmann et al., [3], in order to achieve a decrease in the underlying computational time. We begin this section with the general setup and follow [8]. In the second subsection we switch to [3], describing the difference in the approach compared to [8].

5.1 Basic Pykhtin setup

As before, let us have N different borrowers, one loan per borrower, and we suggest to have them split across K sectors. Let $\bar{L}$ denote the portfolio loss (1) given one-factor credit risk model.

The limiting loss (as $N \to \infty$) can be written using Theorem 3.5. as

$$\bar{L} = \mu(\bar{Y}) = \sum_n w_n \cdot \mathrm{LGD}_n \cdot \mathrm{PD}_n(\bar{Y}) \tag{49}$$

where $\bar{Y}$ represents the single systematic risk factor and

$$\mathrm{PD}_n(y) = \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - s_n y}{\sqrt{1 - s_n^2}}\right). \tag{50}$$

We want to establish connection between one-factor model, in particular between s_n and the multi-factor model's factor loadings r_n. A desirable property would be to relate $\bar{L}$ to the true loss variable L such that

$$\bar{L} = \mathbb{E}[L|\bar{Y}]. \tag{51}$$

Using Proposition 2.5. and the definition (49) we know that

$$\mathrm{VaR}_q(\bar{L}) = \mu(\Phi^{-1}(1-q)). \tag{52}$$

Let the full multi-factor setting be defined by (1) (normalized version with w_n instead of EADs) and (13) with (16). In order to get (51), we first assume that

$$\bar{Y} = \sum_k b_k Z_k \tag{53}$$

and set

$$Y_{k(n)} = \varrho_n \bar{Y} + \sqrt{1 - \varrho_n^2}\epsilon_n \tag{54}$$

where ϵ_n are independent, standard normal, $\sum_k b_k^2 = 1$ to preserve unit variance on $\bar{Y}$ and ϱ_n are chosen such that

$$\varrho_n = \mathrm{Corr}(Y_{k(n)}, \bar{Y}) = \sum_{k=1}^{K} \alpha_{nk} b_k. \tag{55}$$

Having (54), we can rewrite (13) as

$$X_n = r_n \varrho_n \bar{Y} + \sqrt{1 - (r_n \varrho_n)^2}\ \xi_n, \tag{56}$$

where, as before, ξ_n are iid standard normal, independent of $\bar{Y}$. For the conditional expectation this means

$$\mathbb{E}[L|\bar{Y}] = \sum_{n=1}^{N} w_n \cdot \mathrm{LGD}_n \cdot \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n) - r_n \varrho_n \bar{Y}}{\sqrt{1 - (r_n \varrho_n)^2}}\right). \tag{57}$$

Condition (51) holds if and only if the following restriction for the effective factor loadings s_n (recall (49) and (50)) holds

$$s_n = r_n \varrho_n = r_n \sum_{k=1}^{K} \alpha_{nk} b_k \quad \text{for all } n \in \{1, 2, \ldots, N\}. \tag{58}$$

As mentioned before, $\alpha_{nj} = \alpha_{k(n)j}$ for $j = \{1, 2, \ldots, K\}$ and $n \in \{1, 2, \ldots, N\}$.

Assumption 5.1. *In the following we assume that* (58) *holds.*

5.1.1 Choice of b_k, equation (53)

Recall the condition (58). r_n are assumed to be fixed, $\alpha_{ij}, i, j = 1, 2, \ldots, K$ are entries of the lower diagonal Cholesky decomposition matrix of the correlation matrix Θ. But there is a variety of b_k satisfying (58). To minimize the difference between $\mathrm{VaR}_q(\bar{L})$ and $\mathrm{VaR}_q(L)$ we choose b_k in a way to maximize the correlation between $\bar{Y}$ and Y_n for all n. A correlation close to one suggests that the single-factor and the multi-factor model incorporate similar movements, thus dependence on changing state of information. This reduces difference between the quantiles of $\bar{L}$ and L. Mathematically expressed, Pykhtin, [8], suggests to choose $\mathbf{b}^{\mathrm{opt}} = (b_1^{\mathrm{opt}}, b_2^{\mathrm{opt}}, \ldots, b_K^{\mathrm{opt}})$ such that the following maximization problem is solved

$$\mathbf{b}^{opt} = \arg\max_{\mathbf{b}} \sum_{n=1}^{N} \left(\boldsymbol{\alpha}_n \mathbf{b}^T\right) d_n \quad \text{such that} \quad \mathbf{b}\mathbf{b}^T = 1, \tag{59}$$

where $\boldsymbol{\alpha}_n = (\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nK})$. We can solve this problem by introducing Lagrange multiplier via defining

$$\Lambda(\mathbf{b}, \lambda) = \sum_{n=1}^{N} \left(\boldsymbol{\alpha}_n \mathbf{b}^T\right) d_n - \lambda \left(\mathbf{b}\mathbf{b}^T - 1\right).$$

Partial derivations leads to a system of $K+1$ equations

$$\frac{\partial \Lambda}{\partial b_k} = \sum_{n=1}^{N} d_n \alpha_{nk} - 2\lambda b_k \stackrel{!}{=} 0 \qquad \text{for all } k \tag{60}$$

$$\frac{\partial \Lambda}{\partial \lambda} = \sum_{k=1}^{K} b_k^2 - 1 \stackrel{!}{=} 0. \tag{61}$$

Solving (60) for b_k gives b_k^{opt},

$$b_k^{\text{opt}} = \frac{\sum_{n=1}^{N} d_n \alpha_{nk}}{\lambda'} \tag{62}$$

where $\lambda' = 2\lambda$ is chosen such that $\mathbf{b}\mathbf{b}^T = 1$.

Note that the maximization problem introduced additional parameters d_n. After doing empirical tests, [8] concluded that the choice

$$d_n = w_n \cdot \text{LGD}_n \cdot \left(\frac{\Phi^{-1}(\text{PD}_n) - r_n \Phi^{-1}(q)}{\sqrt{1-r_n^2}} \right) \tag{63}$$

is one of the best performing.

5.1.2 Perturbation of the loss variable L

So far we have constructed a random variable $\bar{L}$ such that its quantile $\text{VaR}_q(\bar{L})$ can be calculated analytically and is close to $\text{VaR}_q(L)$. Let us define the difference between $\bar{L}$ and L as a perturbation term $U = L - \bar{L}$. A perturbation variable

$$L_\varepsilon = \bar{L} + \varepsilon U = \bar{L} + \varepsilon (L - \bar{L})$$

describes the scale of perturbation. Now the idea is to expand $\text{VaR}_q(L_\varepsilon)$ in powers of ε around $\bar{L}$ and evaluate at $\varepsilon = 1$. This yields

$$\text{VaR}_q(L) = \text{VaR}_q(\bar{L}) + \frac{d\text{VaR}_q(L_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} + \frac{1}{2}\frac{d^2\text{VaR}_q(L_\varepsilon)}{d\varepsilon^2}\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^3). \tag{64}$$

Let us now investigate each term. We know the analytical expression for the zero order term from (52). The first derivative equals

$$\frac{d\text{VaR}_q(L_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \mathbb{E}[U|\bar{L} = \text{VaR}_q(\bar{L})] = \mathbb{E}[U|\bar{Y} = \Phi^{-1}(1-q)], \tag{65}$$

where we used (52) and (49) to establish second equality. Note that by construction of $\bar{L}$, (51), $E[U|\bar{Y}] = 0$. Hence the first order term in (64) vanishes for any q.

The second order term can be expressed as

$$\frac{d^2\text{VaR}_q(L_\varepsilon)}{d\varepsilon^2}\Big|_{\varepsilon=0} = -\frac{1}{f_{\bar{L}}(\text{VaR}_q(\bar{L}))} \cdot \frac{d}{dl}\left(f_{\bar{L}}(l) \cdot \sigma^2_{U|\bar{L}=l}\right)\Big|_{l=\text{VaR}_q(\bar{L})}, \tag{66}$$

where σ^2 stands for the variance and $f_{\bar{L}}(l)$ is the probability density function of $\bar{L}$, (49). Since $\bar{L} = \mu(\bar{Y})$ is deterministic and monotonically decreasing in $\bar{Y}$, the conditional variances of L and U are equal, i.e.,

$$\sigma^2(y) = \sigma^2_{L|\bar{Y}=y} = \sigma^2_{U|\bar{Y}=y}. \tag{67}$$

Substituting $l = \mu(y)$ in equation (66), we get

$$\frac{d^2 \mathrm{VaR}_q(L_\varepsilon)}{d\varepsilon^2}\Big|_{\varepsilon=0} = -\frac{1}{\phi(\mathrm{VaR}_{1-q}(\bar{Y}))} \cdot \frac{d}{dy}\left(\phi(y) \cdot \frac{\sigma^2(y)}{\mu'(x)}\right)\Big|_{y=\mathrm{VaR}_{1-q}(\bar{Y})}, \tag{68}$$

where $\phi(y)$ is the probability density function of a standard normal random variable. Inserting (68) into equation (64) yields the correction term

$$\begin{aligned}\Delta \mathrm{VaR}_q &= \mathrm{VaR}_q(L) - \mathrm{VaR}_q(\bar{L}) \\ &= -\frac{1}{2\mu'(\mathrm{VaR}_{1-q}(\bar{Y}))} \cdot \left[\frac{d}{dy}\sigma^2(y) - \sigma^2(y)\left(\frac{\mu''(y)}{\mu'(y)} + y\right)\right]\Big|_{y=\mathrm{VaR}_{1-q}(\bar{Y})},\end{aligned} \tag{69}$$

where we used that the first order term is equal to zero and $\phi'(y)/\phi(y) = -y$. For the full derivation of the first and second order terms see [15]. (69) is the key term of this methodology. It adjusts for the multi-factor and single-factor limiting loss distribution difference and also corrects for the finite loan number.

5.1.3 Quantile correction term $\Delta \mathbf{VaR}_q$

In this subsection we derive an explicit formula for the adjustment term $\Delta\mathrm{VaR}_q$, (69). For that we need to determine the conditional mean and variance of the loss L given $\bar{Y} = y$. $\mu'(y)$ and $\mu''(y)$ can be obtained from (49). Since the asset returns X_n are independent conditionally on $\{Z_1, \ldots, Z_K\}$, we can decompose $\sigma^2(y)$ in the following way

$$\sigma^2(y) = \underbrace{\mathrm{Var}[\mathbb{E}[L|Z_1, \ldots, Z_K]|\bar{Y} = y]}_{\sigma^2_\infty(y)} + \underbrace{\mathbb{E}[\mathrm{Var}[L|Z_1, \ldots, Z_K]|\bar{Y} = y]}_{\sigma^2_{\mathrm{Ga}}(y)}. \tag{70}$$

$\sigma^2_\infty(y)$ stands for the variance of the limiting loss distribution (20) conditional on $\bar{Y} = y$. It captures the difference between the multi-factor and the single-factor limiting loss distribution. This fact is accentuated by the result that the first variance term disappears if the single systematic factor $\bar{Y}$ is equal to the independent factors $\{Z_1, \ldots, Z_K\}$.

Theorem 5.2. *The conditional variance term σ^2_∞ is given by*

$$\begin{aligned}\sigma^2_\infty = &\sum_{n=1}^{N}\sum_{m=1}^{N} w_n w_m \mathrm{LGD}_n \mathrm{LGD}_m \cdot \\ &\cdot \left[\mathcal{N}_2(\Phi^{-1}(\mathrm{PD}_n(y)), \Phi^{-1}(\mathrm{PD}_m(y)), \varrho^{\bar{Y}}_{nm}) - \mathrm{PD}_n(y)\mathrm{PD}_m(y)\right]\end{aligned} \tag{71}$$

where $\mathcal{N}_2(\cdot,\cdot,\cdot)$ is a bivariate normal distribution and $\varrho^{\bar{Y}}_{nm}$ is the asset correlation between asset n and m conditional on $\bar{Y}$. Moreover, the derivative of σ^2_∞ is given by

$$\frac{d}{dy}\sigma^2_\infty = 2\sum_{n=1}^{N}\sum_{m=1}^{N} w_n w_m \mathrm{LGD}_n \mathrm{LGD}_m \cdot \mathrm{PD}'_n(y) \cdot \\ \cdot \left[\Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_m(y)) - \varrho^{\bar{Y}}_{nm}\Phi^{-1}(\mathrm{PD}_n(y))}{\sqrt{1-(\varrho^{\bar{Y}}_{nm})^2}} \right) - \mathrm{PD}_m(y) \right], \tag{72}$$

where $PD'_n(y)$ denotes the derivative of $PD_n(y)$ with respect to y.

Remark 5.3. $\varrho^{\bar{Y}}_{nm}$ can be calculated by exploiting equation (13) and the definitions of $\bar{Y}$ and Y_n, in particular

$$X_n = s_n \cdot \bar{Y} + \sum_{k=1}^{K}(r_n\alpha_{nk} - s_n b_k)\cdot Z_k + \sqrt{1-r_n^2}\xi_n.$$

Conditions $s_n = r_N\varrho_n$, (58), and $\sum_k b_k^2 = 1$ yield

$$\varrho^{\bar{Y}}_{nm} = \frac{r_n r_m \sum_{k=1}^{K}\alpha_{nk}\alpha_{mk} - s_n s_m}{\sqrt{(1-s_n^2)(1-s_m^2)}}.$$

The second term, $\sigma^2_{\mathrm{Ga}}(y)$, accounts for the granularity and captures the difference between finite and infinite number of loans in the portfolio. It vanishes for $N\to\infty$, given that $\sum_n w_n^2 \to 0$ a.s.

Theorem 5.4. *$\sigma^2_{Ga}(y)$ is given by*

$$\sigma^2_{Ga}(y) = \sum_{n=1}^{N} w_n \cdot [\mathrm{LGD}_n^2 \cdot [\mathrm{PD}_n(y) \\ - \mathcal{N}_2(\ \Phi^{-1}(\mathrm{PD}_n(y)),\ \Phi^{-1}(\mathrm{PD}_n(y)),\ \varrho^{\bar{Y}}_{nn})]] \tag{73}$$

and its derivative

$$\frac{d}{dy}\sigma^2_{Ga}(y) = \sum_{n=1}^{N} w_n \cdot \mathrm{PD}'_n(y)\cdot(\mathrm{LGD}_n^2 \cdot \\ \cdot [1 - 2\Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_n(y)) - \varrho^{\bar{Y}}_{nn}\Phi^{-1}(\mathrm{PD}_n(y))}{\sqrt{1-(\varrho^{\bar{Y}}_{nn})^2}}\right)]). \tag{74}$$

For the proofs of Theorems 5.2. and 5.4. see [10].

Remark 5.5. The general portfolio model and hence Theorems 5.2. and 5.4. can be easily extended to the case with random LGDs, conditionally independent of X_ns given $Z_1,\dots,Z_K$. For this extension and related proofs see [10]. We assume fixed LGDs in this paper (recall subection 2.1).

Due to the linearity of the adjustment term (69), we can rewrite it in the following way

$$\Delta\mathrm{VaR}_q = \Delta\mathrm{VaR}_q^{\infty} + \Delta\mathrm{VaR}_q^{\mathrm{Ga}}, \tag{75}$$

where each term includes the corresponding conditional variance and its derivative from Theorems 5.2. and 5.4., respectively for $\Delta\mathrm{VaR}_q^{\infty}$ and $\Delta\mathrm{VaR}_q^{\mathrm{Ga}}$.

5.1.4 Expected Shortfall case

Another risk measure, the Expected Shortfall (ES), is also addressed in [8]. As already mentioned (recall subsection 2.2), in contrast to VaR the ES evaluates the average loss that exceeds some fixed level $\mathrm{VaR}_q(L)$, (11).

We know from subsections 5.1.1-5.1.3 how to calculate $\mathrm{VaR}_q(L)$ for any confidence level q. Thus it can be integrated this into the general ES formula (12) and afterwords numerical integration can be applied. Yet [8] suggests to substitute the quantile of the form $\mathrm{VaR}_q(L) = \mathrm{VaR}_q(\bar{L}) + \Delta\mathrm{VaR}_q$ into (12) and this leads to

$$\mathrm{ES}_q(L) = \mathrm{ES}_q(\bar{L}) + \underbrace{\frac{1}{1-q}\int_q^1 \Delta\mathrm{VaR}_s ds}_{\Delta\mathrm{ES}_q}, \tag{76}$$

where the first term is the ES of the comparable one-factor portfolio (established in the beginning of subsection 5.1 and 5.1.1) and the second term is the ES multi-factor adjustment.

Assumption 5.6. *Recall that d_n, equation (63), depend on the level q. This implies the dependence of the factor loadings s_n, (58), on the level q. Pykhtin, [8], suggest to redefine $\{b_k\}_k$ to be equal for all percentile levels above q and, in particular, equal to $\{b_k\}_k$ defined according to equation (62) with the confidence level fixed at q for all greater confidence levels.*

We want to calculate the terms of (76) under the Assumption 5.6. Using (49), we can rewrite (11) as

$$\mathrm{ES}_q(\bar{L}) = \mathbb{E}[\mu(\bar{Y})|\bar{Y} \leq \Phi^{-1}(1-q)] = \frac{1}{1-q}\int_{-\infty}^{\Phi^{-1}(1-q)} \mu(y)\phi(y)dy \tag{77}$$

To find $\Delta\mathrm{ES}_q$, we recall that $\Delta\mathrm{VaR}_q$ equals one half of the second derivative of VaR, (64) (recall that the first derivative vanishes). Using (68), $\Delta\mathrm{ES}_q$ can be rewritten as

$$\Delta\mathrm{ES}_q = -\frac{1}{2(1-q)}\int_q^1 \frac{1}{\phi(y)} \cdot \frac{d}{dy}\left(\phi(y) \cdot \frac{\sigma^2(y)}{\mu'(y)}\right)\Big|_{y=\Phi^{-1}(1-s)} ds, \tag{78}$$

where all terms are defined as in the VaR case. Since (78) is linear in conditional variance $\sigma^2(y)$, $\Delta\mathrm{ES}_q$ can also be represented as the sum of two adjustment parts

$$\Delta\mathrm{ES}_q = \Delta\mathrm{ES}_q^{\infty} + \Delta\mathrm{ES}_q^{\mathrm{Ga}},$$

similarly to VaR case, (75).

5.1.5 Summary

In this section we reviewed analytical approximation methodology for $\mathrm{VaR}_q(L)$ proposed by [8].

We began with the construction of a single-factor model for $\bar{L}$, (49). To closely reflect the multi-factor model (composed of (1), (13) and (16)) in the sense of minimizing the difference between $\mathrm{VaR}_q(L)$ and $\mathrm{VaR}_q(\bar{L})$ we discussed in subsection 5.1.1 special conditions for b_k, (62). Having derived this approximative variable $\bar{L}$, we studied the difference between $\mathrm{VaR}_q(L)$ and $\mathrm{VaR}_q(\bar{L})$. For that we applied Taylor expansion and derived explicit formulas for the first two expansion terms (whereas the first term was shown to vanish by construction). This gave rise to the quantile correction term $\Delta\mathrm{VaR}_q$, (69), which was described in a greater detail in subsection 5.1.3. Having established all necessary results for VaR, we applied them for the ES case. All of this gives us an explicit methodology to analytically approximate credit risk capital determined via VaR, ES or EC for any portfolio, assuming the underlying multi-factor asset return model and the loss variable as in section 5.

5.2 Düllmann et al. modification of Pykhtins approach

[3] suggest a simplified version of [8] with a main motivation to further reduce the computational burden. This is achieved by requiring the input parameters as EAD and PD from (1) only on a sector level. Whereas this sector level PD homogeneity assumption is usually not met in the real life credit portfolios, [3] justify their simplification with numerical examples with good approximation results.

Mathematically expressed, new $\bar{L}$ is defined on a sector level (compare to (49))

$$\bar{L} = \sum_k w_k \cdot \mathrm{LGD}_k \cdot \mathrm{PD}_k(\bar{Y}). \tag{79}$$

where, if one wants to relate it to the loan level case, w_k is the sum of w_n for $n \in$ Sector k and LGD_k is set to the average of LGD_n for $n \in$ Sector k. One can actually interpret the new framework as if having a portfolio consisting of K sectors with one loan per sector and then applying straightforward [8] methodology for this "K loan" portfolio. Thus again the multi-factor VaR is approximated via a VaR for a single-factor model (79) and by adding $\Delta\mathrm{VaR}$, (69).

5.3 Testing Düllmann et al. and Pykhtin methodologies

In this subsection the impact off the simplifying assumption of [3], discussed in subsection 5.2, is studied. We want to compare the resulting $\mathrm{VaR}(L)$ approximation and in particular its building blocks, i.e., the one factor approximation term $\mathrm{VaR}(\bar{L})$ and the adjustment terms $\Delta\mathrm{VaR}^\infty$ and $\Delta\mathrm{VaR}^{\mathrm{Ga}}$, (75), obtained either using purely [8], subsection 5.1, or [3]. The reference result in all simulations is based on 2 million Monte-Carlo sample estimate, which we call crude

Monte-Carlo estimate.

Let the loss variable be modelled as in (1) and choose the asset returns and systematic risk factors models given by (13) and (16), respectively. Let us assume that there are 12 sectors, $K = 12$, and assume each portfolio consists of $N = 1200$ loans, thus 100 loans per sector. For the needs of experiment, 4 inter-sector correlation matrices $\Theta_1, \dots, \Theta_4$ are constructed with entries describing correlation of the 12 sector performance indices. This, in particular, implies ones on diagonal, i.e., $\theta_{ii} = 1$ for every $i = 1, 2, \dots, K$, and that every entry is bounded by one in its absolute value. Every Θ_i is also symmetric and positive-definite.

We begin with one portfolio, simulated as in Example 2, subsection 4.6, with (45) and normalize EAD, (5). This portfolio is coupled with 4 different inter-sector correlation matrices `nCorr`, `lCorr`, `mCorr` and `hCorr` from Example 5, and the $\mathrm{VaR}_{0.999}$ is estimated using Pykhtin methodology, [8], Dülmmann et al, [3], and crude Monte-Carlo based on 2 million samples. Results are summarized in Table 1.

Table 1: Negative effect of sector level aggregation in Düll. Note that the zero order term approximation $\bar{L}$, (52), is the same under Düll and Pykh. The adjustment terms are calculated using equation (69) and Theorems 5.2. and 5.4. for $\Delta\mathrm{VaR}^{\infty}$ and $\Delta\mathrm{VaR}^{\mathrm{Ga}}$, respectively.

	Method	$\mathrm{VaR}(\bar{L})$	$\Delta\mathrm{VaR}^{\infty}$	$\Delta\mathrm{VaR}^{\mathrm{Ga}}$	$\mathrm{VaR}(L)$, (69)
`nCorr`	Pykh	0.0733	0.0048	0.0157	**0.0939**
	Düll	0.0729	0.0056	0.2934	0.3719
	crude MC				**0.0930**
`lCorr`	Pykh	0.0790	0.0044	0.0140	**0.0975**
	Düll	0.0790	0.0053	0.2602	0.3445
	crude MC				**0.0972**
`mCorr`	Pykh	0.1760	5.6316e-04	0.0072	**0.1838**
	Düll	0.1792	8.2260e-04	0.1359	0.3160
	crude MC				**0.1845**
`hCorr`	Pykh	0.2195	1.1703e-05	0.0065	**0.2260**
	Düll	0.2249	9.4272e-05	0.1217	0.3467
	crude MC				**0.2262**

Let us first comment on the performance of Düllmann et al. approach. It certainly provides incorrect granularity adjustment term. Intuitively this is clear since the term $\Delta\mathrm{VaR}^{\mathrm{Ga}}$ captures the difference between the finite and infinite number of loans in the portfolio. Thus, consolidating loans of each sector into an average representative with an exposure size $\mathrm{EAD}_k = \sum_{j \in \mathrm{Sector\ k}} \mathrm{EAD}_j$ harms the calculation, since in this case the granularity adjustment term basically responds to a K loan portfolio, where $K << N$. As simulations showed, Düllmann tends to overestimate $\Delta\mathrm{VaR}^{\mathrm{Ga}}$ compared to Pykhtin and their relative difference decreases as the number of sectors K increases. In figure 14 a fixed portfolio with varying sector classifications (resulting in different total numbers of sectors K) is considered. Plot values are normalized with respect to Pykhtin results, yielding constant value 1 for Pykhtin and showing the relative granularity adjustment difference for Düllmann.

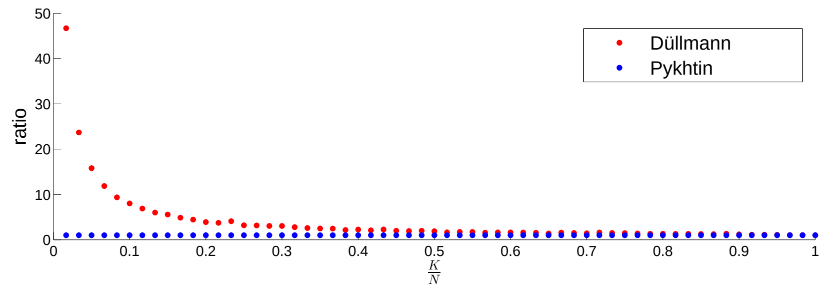

Figure 14: The relative difference of granularity adjustment $\Delta\text{VaR}^{\text{Ga}}$ obtained with Düllmann and Pykhtin approaches. Plot values are normalized with respect to Pykhtin results individually for each $\frac{K}{N}$. Clearly ratio decreases as the number of sectors K increases since sector aggregated portfolio converges to the actual constellation.

In further experiments we use Düllamm approach but with the granularity adjustment on a loan level. Such approach is also suggested by [12]. We abbreviate this methodology as PykhDüll.

Considering crude MC $\text{VaR}(L)$ results as a reference, we see that Pykhtin provides good results (bold results, table 1). Even more, since crude MC results may slightly change from simulation to simulation (estimation variance due to different random number sequences during numerical sampling), one can get crude MC results even equal to Pykhtin. Another observation is the decreasing size of the $\Delta\text{VaR}^{\infty}$ as the overall dependence increases (expressed by Θ_i). Intuitively this is expected, since having almost no correlation, `lCorr`, or no correlation, `nCorr`, makes it difficult for a single systematic risk factor $\bar{Y}$, (53), to capture and recreate the multi-factor dynamics. This, in its turn, implies larger $\Delta\text{VaR}^{\infty}$ terms. Vice versa, for highly correlated systematic risk factors, an appropriate choice of b_k, (62), makes $\bar{Y}$ a better representative of the general setup and thus implies smaller $\Delta\text{VaR}^{\infty}$ terms.

5.4 Summary

In this section VaR or ES analytical approximation methodology [8] and its simplification [3] was presented. Based on several test portfolios, we argued against using sector level granularity adjustment in context of this thesis. As a computational time versus precision dilemma consensus, a mix of a sector level $\Delta\text{VaR}^{\infty}$ and a loan level $\Delta\text{VaR}^{\text{GA}}$ was suggested. The corresponding approach was denoted by PykhDüll.

6 Comparison study of Cespedes and Düllmann approaches

Although the name of the section states the comparison of methodologies presented in [1] and [3], we replace [3] by a symbiosis of [8] and [3] due to the discussion of subsection 5.3. Sector level aggregation, introduced in [3], does decrease computational time, but its application on a particular portfolio construction, frequently used in this thesis, provided poor results. Examples are presented in Table 1.

6.1 Performance test

For comparison purposes we construct one portfolio $\hat{P}$ consisting of $K = 12$ sectors and 100 loans per sector. We consider model (16) for the systematic risk and choose $\mathrm{EC}(L) = \mathrm{VaR}_{0.999}(L) - \mathbb{E}[L]$ as the risk measure defining risk capital. Let the risk parameters be chosen as follows:

- $\mathrm{PD}_n \sim \mathcal{U}(0.03, 0.06)$ for every loan n,
- $\mathrm{EAD}_n \sim \mathcal{U}(10, 90)$ for every loan n,
- $\mathrm{LGD}_n \sim \mathcal{U}(0.3, 0.8)$ for every loan n
- and $r_k \sim \mathcal{U}(0.4, 0.6)$ for every sector k.

This choice is based on the same reasoning as in Example 1, subsection 4.6. We take the inter-sector correlation matrices $\Theta_1, \ldots, \Theta_4$ already used in Example 5. of subsection 4.6 and in subsection 5.3, also called as `nCorr`, `lCorr`, `mCorr` and `hCorr` (described in Appendix A). This results in the following average correlation $\bar{\beta}$ (42):

- `nCorr` $\Rightarrow \bar{\beta} = 0$,
- `lCorr` $\Rightarrow \bar{\beta} = 0.0159$,
- `mCorr` $\Rightarrow \bar{\beta} = 0.4264$,
- `hCorr` $\Rightarrow \bar{\beta} = 0.7177$.

We manipulate with loan exposure sizes from $\hat{P}$ to get 4 different portfolios in terms of relative sector exposures, thus making some sectors dominant. The reason for this is to construct portfolios which show wide range of CDI values. We call the resulting portfolios lCDI, mlCDI, mhCDI and hCDI. For the definitions and intuitive interpretations of the names let us take `lCorr` as the inter-sector correlation matrix. This then leads to:

- lCDI $\Rightarrow$ CDI $= 0.0844$. Name stands for “low” CDI. lCDI is equal to $\hat{P}$. $1/\mathrm{CDI} \approx 11.8$.

- mlCDI $\Rightarrow$ CDI $= 0.1208$. Name stands for "medium-low" CDI. mlCDI constructed from $\bar{P}$ by taking $20 \cdot \mathrm{EAD}_n$ for all n from sectors $k = 1, \ldots, 8$. $1/\mathrm{CDI} \approx 8.3$.
- mhCDI $\Rightarrow$ CDI $= 0.1842$. Name stands for "medium-high" CDI. mhCDI constructed from $\bar{P}$ by taking $30 \cdot \mathrm{EAD}_n$ for all n from sectors $k = 1, \ldots, 5$. $1/\mathrm{CDI} \approx 5.4$.
- hCDI $\Rightarrow$ CDI $= 0.4775$ Name stands for "high" CDI. hCDI constructed from $\bar{P}$ by taking $200 \cdot \mathrm{EAD}_n$ for all n from sectors $k = 1, 2$. $1/\mathrm{CDI} \approx 2.1$.

Thus we have 4 portfolios and 4 inter-sector correlation matrices, which gives 16 portfolio combinations capturing wide spread of CDI and $\bar{\beta}$ values. Note that after each manipulation with exposure sizes, we renormalize EAD_n to w_n, (5).

The visualization of the different nature of portfolios is presented in figure 15 and 16. In figure 15 we plot the loss probability density functions for lCDI

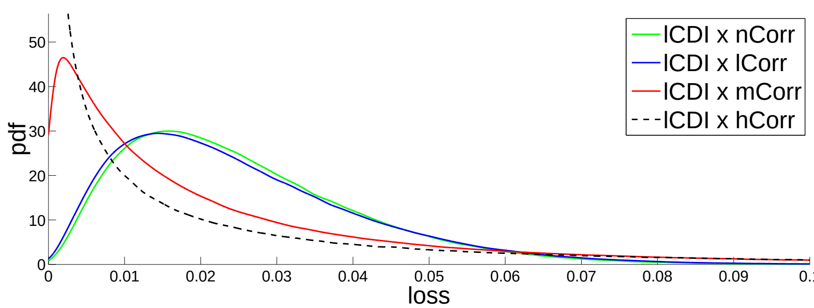

Figure 15: Approximation of the loss probability density functions of L, (1), for different portfolio constellations. Higher average correlation across sectors increases the probability of small losses and shifts more probability mass to the tails.

under 4 different inter-sector correlation matrices. It is shown, that the density peak is shifted to the left as the underlying average correlation increases and in the same time more probability mass is shifted to the right tails. Whereas in figure 16 we observe how increase in average sector correlation can be compensated by a more homogeneous loan exposure size distribution. In both plots it is clear that we deal with skewed probability densities and heavy tails. In particular, heavy tails point at a significant difference in using VaR or ES, since the loss of L exceeding some VaR level may vary extremely. This is one of the main arguments for choosing ES instead of VaR as a risk measure in the definition of credit risk capital. This wide extreme value spread for L also motivates for variance reduction techniques for credit risk capital (or equivalently the loss measure) Monte-Carlo estimates, shortly discussed in section 7. Yet in our calculations we stay with VaR, which is, first of all, widely used in the industry (partly as a historical heritage and also due to simple statistical estimation)

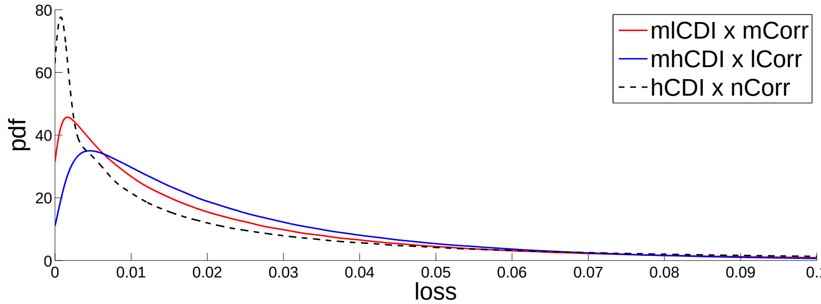

Figure 16: Approximation of the loss probability density functions of L, (1), for different portfolio constellations. Homogeneously distributed loan exposure sizes across sectors partly compensate increased correlation across sectors. Small losses for a portfolio with two dominant sectors (hCDI) are still more likely to occur even when the underlying sector dynamics are linearly independent (`nCorr`).

and, secondly, is at the core of credit risk management both in Basel II and III, [4] and [5] respectively.

Finally, we present the resulting Table 2, where crude MC based on 2 million samples is compared to in subsection 5.3 introduced symbiosis of [8] and [3], abbreviated as PykhDüll, and [1] results. DF parametrization as in (48), Example 6, subsection 4.6. 2 million sample crude MC estimations are consid-

Table 2: Comparing PykhDüll and Düll. EC = $\mathrm{VaR}_{0.999}$ - $\mathbb{E}[L]$. Most of the Cesp results can be cosnidered as poor. Crude MC is the reference.

	Method	**hCDI**	**mhCDI**	**mlCDI**	**lCDI**
`hCorr`	crude MC	0.2017	0.1742	0.1747	0.1754
	PykhDüll	0.2018	0.1736	0.1752	0.1734
	Cesp	0.2010	0.1731	0.1745	0.1720
`mCorr`	crude MC	0.2052	0.1454	0.1157	0.1253
	PykhDüll	0.2047	0.1459	0.1142	0.1249
	Cesp	0.2030	0.1435	0.1043	0.1132
`lCorr`	crude MC	0.1368	0.0709	0.0544	0.0449
	PykhDüll	0.1340	0.0693	0.0534	0.0441
	Cesp	0.1506	0.0739	0.0551	0.0452
`nCorr`	crude MC	0.1381	0.0672	0.0517	0.0412
	PykhDüll	0.1347	0.0655	0.0503	0.0405
	Cesp	0.1510	0.0705	0.0532	0.0427

ered as reference results. As Table 2 shows, most of the EC estimations based on DF (48) can be interpreted as poor. More precisely, risk capital is overestimated for relatively low average correlation portfolios, i.e., `lCorr` and `nCorr` portfolios. Whereas it is mostly overestimated when the underlying sector dependence structure given by `mCorr` and `hCorr`. One could consider this as a serious drawback since less diversified portfolios involve higher risks and thus their underestimation is particularly undesirable. Note that due to exposure size normalization (5), EC is a ratio of the total portfolio. Thus, for instance, 0.001 increase in EC for a 100 million US dollar portfolio implies additional 100′000 US dollar risk capital, that a financial institution is obliged to put aside as a risk buffer.

Clearly one can reach higher estimation precision using [1], when the underlying DF parametrization stems from sampled portfolios, which are close to the portfolio of interest, as discussed in Example 5, subsection 4.6. An example is presented in Table 3, where we calculate DF based on only 5′000 portfolios, which all follow (16) systematic risk model with inter-sector correlation matrix `nCorr`, and using 100′000 samples for multi-factor VaR estimation of each portfolio. But this precision is mostly achieved by knowing the prospective real life portfolio configurations and also includes prediction of the inter-sector correlations. If a company is able to describe its prospective portfolios in such a detail, [1] can be a good risk management tool. As for PykhDüll, in most of the cases

Table 3: EC = $\mathrm{VaR}_{0.999}$ - $\mathbb{E}[L]$. DF, (44), parametrization based on the modell (16) `nCorr` data (yellow dots in figure 11). Obviously DF, that was suited for a particular portfolio constellation, shows better precision.

	Method	**hCDI**	**mhCDI**	**mlCDI**	**lCDI**
`nCorr`	crude MC	0.1381	0.0672	0.0517	0.0412
	PykhDüll	0.1347	0.0655	0.0503	0.0405
	Cesp	0.1503	0.0686	0.0512	0.0407

it performed better than Cesp. A good sign is that it showed higher precision for heavy tailed portfolios. Of course, the main advantage of PykhDüll is its generality applicability. With regards to computational requirements for a given DF parametrization Cesp is much faster[3] than PykhDüll. Yet to once obtain DF itself, serious computational time needs to be invested.

6.2 Summary

To test the performance of PykhDüll and Cesp, 16 portfolios were created covering a range of CDI and $\bar{\beta}$ values. DF parametrization was based on resuts of Example 6, subsection 4.6. A crude MC approach was considered as a reference. In most of the cases Cesp showed poor estimation quality. We again pointed at the importance of "reasonable" DF parametrization scheme in the sense that one should avoid blind usage of DF for any portfolio constellation,

[3]Based on Matlab codes written for the purposes of this thesis.

as discussed in context of table 3. Yet the relative ordering of credit risk values for all portfolios under Cesp, [1], is consistent with PykhDüll and crude MC. Since Cesp is the fastest among 3 methods, it can be considered as a good risk management tool for fast evaluation purposes.

7 Extensions

As discussed in section 4, Cespedes et al [1] methodology relies on Monte-Carlo estimations. This approach can be immediately justified by the fact that in general crude Monte-Carlo result is being used as a reference for any semi-analytic or analytic credit risk capital approximation technique. The problem lies in the time consuming crude MC calculations when many thousand portfolios need to be considered. This forces a compromise between the MC sample size, number of portfolios and loans. We have seen the impact of changing MC sample size in Example 4 of subsection 4.6. Even more, high percentile level VaR or ES require accurate estimation of low probability events of large losses. But the normality assumption for the asset returns (13) and the corresponding dependence mechanism makes it difficult to simulate such rare events. Furthermore, it is difficult to simulate large losses for well diversified high-rated portfolios, i.e., low probablity events. Additionally to that, small sample size MC estimations show high variance[4]. This motivates to either consider some variance reduction techniques or to apply "brute" force - high performance computing. Several variance reduction techniques as, for instance, anthithetic sampling or control variates can be found in the literature, see [23] or [21]. One may also consider low discrepancy sequences and quasi Monte-Carlo methods, [22], to improve Monte-Carlo performance in some applications. Yet our problem framework is well suited for another popular variance reduction technique, called importance sampling.

In subsection 7.1 we introduce importance sampling technique, as by Glasserman and Li, [16], and provide related examples in subsection 7.2. We briefly discuss high performance computing in subsection 7.3.

7.1 Importance sampling

In this subsection we present a two step importance sampling procedure for the loss variable (1) under the systematic risk model (16).

In order to statistically estimate high percentile VaR or ES of a random variable, it is good to have its samples and corresponding probabilities both above and below the actual VaR level. Denote by p_x the Monte-Carlo estimation of $P(L \geq x)$. Then we can decompose the variance of p_x to

$$\mathrm{Var}(p_x) = \mathbb{E}[\mathrm{Var}(p_x|\mathbf{Z})] + \mathrm{Var}(\mathbb{E}[p_x|\mathbf{Z}]), \tag{80}$$

where $\mathbf{Z} = (Z_1, \ldots, Z_K)$ from (16). To sample larger values of L, (1), we manipulate the sampling $\mathbf{Z}$ and additionally transform the conditional default probabilities $\mathbf{PD}_n$, which is a vector of conditional default probabilities PD_n, (18).

We begin with $\mathrm{Var}(p_x|\mathbf{Z})$ from (80). Conditional on $\mathbf{Z}$, the asset returns (13) are independent. To improve p_x estimation, we want to increase the default

[4] Note that the random number seed is reshuffled in every calculation.

probabilities. For that we apply an exponential transformation

$$\mathrm{PD}_{n,\theta} = \frac{\mathrm{PD}_n \cdot e^{\theta\ \mathrm{EAD}_n \mathrm{LGD}_n}}{1 + \mathrm{PD}_n (e^{\theta\ \mathrm{EAD}_n \mathrm{LGD}_n} - 1)} \tag{81}$$

for some $\theta \geq 0$. If $\theta > 0$ then a larger related loss $\mathrm{EAD}_n \mathrm{LGD}_n$ indeed increases the default probability. We compensate the change in default probabilities with a likelihood ratio

$$\exp(-\theta L + \psi(\theta)), \tag{82}$$

where

$$\psi(\theta) = \sum_{n=1}^{N} \log(1 + \mathrm{PD}_n (e^{\theta\ \mathrm{EAD}_n \mathrm{LGD}_n} - 1)) \tag{83}$$

is the cumulant generating function of L. For any θ,

$$\mathbb{1}_{\{L \geq x\}} \exp(-\theta L + \psi(\theta)) \tag{84}$$

is an unbiased estimator of p_x for L generated using $\mathrm{PD}_{n,\theta}$. We choose θ such to reduce $\mathrm{Var}(p_x|\mathbf{Z})$. Minimizing variance is equivalent to minimizing the second moment, which can be bounded as

$$\mathbb{E}_\theta \left[\mathbb{1}_{\{L \geq x\}} \exp(-2\theta L + 2\psi(\theta)) \right] \leq \exp(-2\theta x + 2\psi(\theta)) \quad \text{for all } \theta \geq 0. \tag{85}$$

To minimize the bound, we do maximization of $\theta x - \psi(\theta)$ over $\theta \geq 0$. The function ψ is strictly convex and passes through the origin, thus we choose

$$\theta_x = \max\{0, \text{unique solution to } \psi'(\theta) = x\}. \tag{86}$$

Note that θ_x depends on the initial choice of x. In practice x is set to be a quick crude MC estimation of $\mathrm{VaR}_q(L)$ based on a small pre-defined number of samples. To justify this, numerical simulations show good estimation results of $P(L \geq y)$ for y greater than x, as by [16].

We switch to the second term of (80), which is $\mathbb{E}[p_x|\mathbf{Z}] = P(L \geq x|\mathbf{Z})$ due to (84). So we are interested in choosing a distribution for $\mathbf{Z}$ which would reduce the variance of estimating $P(L \geq x|\mathbf{Z})$. Idealistically one would sample $\mathbf{Z}$ from the density proportional to

$$z \mapsto P(L \geq x|\mathbf{Z}) e^{-z^T z/2} \tag{87}$$

since this would imply zero variance. However to make this a density, the normalization constant is the value $P(L \geq x)$ which we actually seek. [16] and other authors suggest to sample $\mathbf{Z}$ from a multivariate normal distribution with mean

$$\mu = \arg\max_{\mathbf{z}} P(L \geq x|\mathbf{Z}) e^{-\mathbf{z}^T \mathbf{z}/2}. \tag{88}$$

Optimal with respect to (88) μ can be numerically approximated in the following way. Define

$$F_x(\mathbf{z}) = -\theta_x(\mathbf{z}) x + \psi(\theta_x(\mathbf{z}), \mathbf{z}) \tag{89}$$

where $\psi(\theta_x(\mathbf{z}), \mathbf{z})$ denotes (83) with $PD_n(\mathbf{z})$ for all loans n, and $\mathbf{z}$ is a sample of $\mathbf{Z}$. The inequality $\mathbb{1}_{\{y>x\}} \leq \exp(\theta(y-x)), \theta \geq 0$ yields

$$P(L \geq x|\mathbf{Z}=\mathbf{z}) \leq \mathbb{E}\left[\exp(\theta_x(\mathbf{Z}))(L-x)|\mathbf{Z}=\mathbf{z}\right] = \exp(F_x(\mathbf{z})). \tag{90}$$

Using this bound for (88) and taking logarithms yields optimization problem

$$\hat{\mu} = \arg\max_{\mathbf{z}} \left\{ F_x(\mathbf{z}) - \frac{1}{2}\mathbf{z}^T\mathbf{z} \right\}, \tag{91}$$

where $\hat{\mu}$ is an approximation for the IS mean μ.

For related asymptotical optimality proofs and conditions see [16] or [24]. In general asymptotic efficiency has been shown only under certain portfolio conditions.

Finally, the full scheme for $\mathrm{VaR}_q^{\mathrm{mf}}$ estimation is:

1. Sample $\mathbf{z}$ from $\mathcal{N}_{\dim(\hat{\mu})}(\hat{\mu}, I)$, where I is the identity matrix.
2. Compute $\theta_x(\mathbf{z})$ using (86) and the twisted conditional default probabilities $\mathrm{PD}_{n,\theta_x(\mathbf{z})}$ as in (81) for every loan n.
3. Sample L under the twisted conditional distribution and the corresponding likelihood ratio
$$\exp(-\theta L + \psi(\theta)) \exp(-\hat{\mu}^T z + \hat{\mu}^T \hat{\mu}/2). \tag{92}$$

After sampling $L_1, L_2, \ldots, L_V$ for some $V \in \mathbb{N}$ and the corresponding likelihoods $W_1, W_2, \ldots, W_V$, order $\{L_i\}_{i=1}^V$ as $L_{(1)} \leq L_{(2)} \leq \ldots \leq L_{(V)}$. Now find

$$V^* = \sup\left\{ v : \frac{1}{V} \sum_{i=v}^{V} W_{(i)} \geq 1-q \right\} \tag{93}$$

and return the $\mathrm{VaR}_q^{\mathrm{mf}}$ estimator $L_{(V^*)}$. Said in words, the right tail of the ordered likelihoods $\{W_{(i)}\}_{i=1}^V$ is of the smallest size. We begin with $W_{(V)}$ and keep on adding $W_{(V-1)}$ and so on as long as the sum becomes greater than $1-q$. At this point we stop, fix the index of the last added $W_{(V^*)}$ and return the corresponding $L_{(V^*)}$. For the statistical ES estimator, based on sample data, we choose [24, Eq. 5].

7.2 IS performance test

In this subsection IS performance results are presented. Calculations are based on 4 portfolios taken from subsection 6.1. Three percentile levels $q = 0.95$, 0.99, 0.999 are considered. In all following tables crude MC_1 stands for crude Monte-Carlo estimation based on $100'000$ samples, crude MC_2 based on 1 million samples, IS_1 is the IS estimation based on $100'000$ samples and IS_2 denotes the IS estimation based on $20'000$ samples. Additionally to that, let IS_3 denote the importance sampling procedure based on $100'000$ samples, but without probability transformation, e.g., omit step 2. in $\mathrm{VaR}_q^{\mathrm{mf}}$ estimation scheme from subsection 7.1 and replace (92) by

$$\exp(-\hat{\mu}^T z + \hat{\mu}^T \hat{\mu}/2). \tag{94}$$

100 runs of each estimation procedure is performed and the mean and standard deviation (std) for both VaR and ES results are listed. For the IS_i, $i = 1, 2, 3$, the first 10% of the samples were used to calculate the initial estimator of VaR_q which is the x from subsection 7.1.

Table 4: $q = 0.95$. VaR and ES statistics, based on 100 repetitions. Crude MC_1 precision competitive with importance sampling (IS) for moderate quantile levels (if compared to $q = 0.999$, table 5). Portfolios defined in subsection 6.1.

portfolio config.	Method	VaR_q mean	VaR_q std	ES_q mean	ES_q std
hCDI & `hCorr`	crude MC_1	0.0920	4.7283e-04	0.1274	7.0696e-04
	IS_1	0.0921	1.4850e-04	0.1276	1.5054e-04
	IS_2	0.0921	3.6170e-04	0.1276	3.2644e-04
	IS_3	0.0921	1.8361e-04	0.1276	1.4707e-04
mhCDI & `mCorr`	crude MC_1	0.0718	3.1372e-04	0.0971	5.0507e-04
	IS_1	0.0718	1.2711e-04	0.0970	9.9494e-05
	IS_2	0.0718	2.7586e-04	0.0970	2.2059e-04
	IS_3	0.0718	1.3012e-04	0.0970	9.9649e-04
mlCDI & `lCorr`	crude MC_1	0.0477	1.1935e-04	0.0563	1.4272e-04
	IS_1	0.0477	1.6023e-04	0.0563	1.3569e-04
	IS_2	0.0476	3.4411e-04	0.0562	3.2871e-04
	IS_3	0.0477	2.2479e-04	0.0562	2.0418e-04
lCDI & `nCorr`	crude MC_1	0.0437	8.7801e-05	0.0499	1.2348e-04
	IS_1	0.0437	2.0667e-04	0.0499	6.9718e-04
	IS_2	0.0437	8.3421e-04	0.0499	8.4528e-04
	IS_3	0.0437	2.7309e-04	0.0498	6.2491e-04

As tables 4 and 5 show, for less diversified portfolios variance reduction technique [16] provided increased estimation quality, whereas this is not the case for better diversified portfolios as mlCDI & `lCorr` and lCDI & `nCorr`. Clearly the relative difference in mean VaR and ES between IS and crude MC is higher when considering higher level quantiles, i.e., higher q level. The main drawback is the increased computational time, even compared to crude MC_2. Note that these calculations are portfolio-wise. The time consuming computations based on importance sampling could be replaced by a simpler, brute force approach, namely increased number of simulated portfolios in some region of interest using crude MC based on more samples. Large number of artificial portfolios and the non-linear regression could average out the error introduced by estimation variance. It seems that from practical point of view the brute force approach is more favourable.

Table 5: $q = 0.999$. VaR and ES statistics, based on 100 repetitions. Estimation variance increases with q and for portfolios with more probability mass in the tails. Increased number of samples reduces estimation variance both for crude Monte-Carlo and in importance sampling methods. IS performs better for less diversified portfolios. Considerable decsrease in computational time is achieved by dropping probability transformation (step 2, subsection 7.1), IS_3, whereas the precision stays close to the full scheme IS_1. Portfolios defined in subsection 6.1.

portfolio config.	Method	VaR_q mean	VaR_q std	ES_q mean	ES_q std
	crude MC_1	0.2303	0.0037	0.2641	0.0046
	crude MC_2	0.2300	9.6624e-04	0.2619	0.0013
hCDI & `hCorr`	IS_1	0.2300	2.2492e-04	0.2616	1.8232e-04
	IS_2	0.2299	5.7512e-04	0.2613	0.0022
	IS_3	0.2299	2.7987e-04	0.2616	1.9329e-4
	crude MC_1	0.1705	0.0026	0.1958	0.0033
	crude MC_2	0.1704	6.9944e-04	0.1942	8.7569e-04
mhCDI & `mCorr`	IS_1	0.1703	1.8958e-04	0.1941	1.6196e-04
	IS_2	0.1704	5.3229e-04	0.1941	5.0803e-04
	IS_3	0.1704	2.3001e-04	0.1941	1.6253e-04
	crude MC_1	0.0794	7.8185e-04	0.0874	0.0010
	crude MC_2	0.0795	2.8682e-04	0.0868	3.6309e-04
mlCDI & `lCorr`	IS_1	0.0786	0.0016	0.0841	0.0063
	IS_2	0.0783	0.0022	0.0833	0.0064
	IS_3	0.0784	0.0016	0.0818	0.0097
	crude MC_1	0.0666	5.3899e-04	0.0724	6.7513e-04
	crude MC_2	0.0667	1.6454e-04	0.08718	2.0314e-04
lCDI & `nCorr`	IS_1	0.0667	8.2234e-04	0.0704	0.0059
	IS_2	0.0666	0.0016	0.0685	0.0086
	IS_3	0.0668	0.0017	0.0694	0.0082

Table 5 shows that IS_3 leads to similar estimation quality as in the case of the full importance sampling scheme IS_1. Nevertheless a considerable reduction in computational time speaks in favour of IS_3. Furthermore, IS_3 uses approximately the same computational time as IS_2 and even provides better results. One can notice that importance sampling procedures as such show better performance in less diversified portfolios, thus for heavy tailed loss distributions L, (1), where IS can benefit from more synchronized movements in the underlying systematic risk factors and capture this behaviour via sampling distribution mean shifting, as given by (91). This motivates to apply both IS_3 and crude MC_3 in one DF parametrization procedure, where the usage of one or another depends on the CDI and $\bar{\beta}$ of the artificial portfolio, which is being processed.

7.3 High performance computing

7.3.1 Discussion on high performance computing

In the last five decades financial instruments have expanded both in their complexity and size. The associated risks need to be handled quickly although it gets harder to identify them. Alongside with the mathematical studies one can also profit from new computation abilities. We exploited the advantages of the ETHZ supercomputer Brutus when performing DF parametrization. Note that the scheme, presented in subsection 4.3, is well suited for independent parallelization. Steps 3.-10. can be simulated in parallel. The resulting data of all parallel computations is collected and used for step 11. Additionally to that, sectorization of portfolio loans in order to capture the underlying dependence structures under the available market data frameworks also reduces problems dimensionality. As by Longstaff and Schwartz (2001), [18], this simple parallelization is close to an optimal approach in practice. Yet the speed of each Monte-Carlo simulations as such, which is at the core of [1], can be improved by using, for instance, GPUs, as by Spiers and Wallez (2010), [17]. But this implies additional software challenges and a consideration whether some particular problem's framework and data gathering will not counter-react and actually reduce achieved computational speed-up in terms of hardware.

7.3.2 Note on ETH Zürcih central cluster Brutus

Brutus is the central high-performance cluster of ETH Zürich. It is jointly owned by nearly 50 professors from 12 departments and the IT Services, who are responsible for the acquisition and management of the system. The part financed by the IT Services is made available to the whole scientific community of ETH. Brutus is a heterogeneous system containing 11 kinds of standard, large-memory, GPU and legacy compute nodes. The peak performance of Brutus is slightly over 200 teraflops ($200 \cdot 10^{12}$ floating-point operations per second). Brutus directly allows a wide range of applications, in particular, third-party applications such as MATLAB, ANSYS CFX, ANSYS FLUENT and others. All researchers from ETH Zürich are allowed to use Brutus without restriction. It was ranked the 88^{th} fastest computer in the world in November 2009 (`top500.org`) and showed to be the most energy efficient general purpose supercomputer in the world at that time (`Heise.de`).

7.4 Summary

Reliance of [1] on Monte-Carlo estimations and the difficulties of estimating high level quantiles for heavy tailed random variables motivated to consider improved Monte-Carlo methods, particularly importance sampling. Subsection 7.1 introduced IS theoretical background. We performed test for a variety of IS and crude MC constellations in subsection 7.2. The main IS drawback is the considerable increase in computational time due to the many additional optimization operations, step 2, subsection 7.1. Yet the increased precision can also be achieved by a crude MC based on increased number of samples, which can still be faster. Furthermore, for well-diversified portfolios crude MC showed better estimation quality. Additionally to that, densely sampled portfolios and non-linear regression may cancel out error, introduced in each portfolio's $\mathrm{VaR}^{\mathrm{mf}}$ estimation. As a final remark, we suggest to use crude MC in case one wants to avoid the additional implementational burden or to apply a mix of crude MC and IS_3 as discussed at the end of subsection 7.2.

8 Conclusion

In this thesis a comparison of two methodologies for credit risk capital estimation is presented. Each bank is interested in providing safe services to its costumers and investors. Nevertheless, a bank is aimed at maximizing its profits and thus exploiting all financial resources at its disposal, leaving less as a safety buffer capital. A trade-off has to be considered. This implies, first of all, a discussion on a reasonable credit risk measure and, secondly, an appropriate calculation schemes. In section 2 we introduced and discussed typical risk measures. After that a general credit portfolio loss model was presented in section 3. Then two methodologies for credit risk capital estimation are revisited. Understanding some limitations and a comparison of the two methodologies is the goal of the thesis.

If one has to deal with a single portfolio or rarely upcoming calculations, PykhDüll method ([3] with the granularity adjustment on a loan level) is a good choice. So far it is the only know method in the literature to provide analytical estimates of VaR and ES under the multi-factor asset return framework. It showed good results for all considered types of portfolios. The main advantage of Pykhtin approach is its universality. It also provides an opportunity to monitor the evolution (after adding or removing some loans or due to the changes in the underlying risk parameters of one or more borrowers) of a portfolio in terms of $\Delta\mathrm{VaR}^{\infty}$ and $\Delta\mathrm{VaR}^{\mathrm{Ga}}$, which capture the difference to the infinitely fine grained ASRF portfolio.

In contrast to PykhDüll, Cespedes methodology, [1], explicitly includes three risk management instruments. It provides fast portfolio comparison in terms of two diversity capturing measures as CDI, (37), and $\bar{\beta}$ (β from (14), (41), (42)), and the risk measure itself once the DF, (44), is obtained. Yet it is important to appropriately set up the DF parametrization procedure, an example of which is presented in subsection 4.4. This is the main drawback of [1], since if some portfolio of interest falls into the region where none or small number of artificial portfolios were considered, the results are unreliable. Furthermore, adjusting DF by adding new artificial portfolios will not only affect the resulting parametrization formula and thus could make all results based on previous parametrization not fully consistent with the new ones, but is also time consuming. Another drawback is the inability of CDI and $\bar{\beta}$ to fully diversify credit portfolios. As discussed in subsection 4.5, CDI does not capture name concentration risks, whereas $\bar{\beta}$ may not distinguish two portfolios of the same size with different underlying inter-sector dependence structure. Apart from that, the model for DF parametrization, step 11, subsection 4.3, is an issue itself. The motivation for the polynomial type DF parametrization lies in the Taylor expansion of (38). This is even more important under such artificial portfolio sampling frameworks, which result in concentrated representation of some small part of CDI-$\bar{\beta}$ plane. We have seen this under the asset returns model (16) in Example 5, subsection 4.6. In this case it could happen that some more sophisticated parametrization models perform better, i.e., in the sense of data fit statistics as adjusted R^2. These models may lead to distorted credit risk capital results when applying them to portfolios with CDI and $\bar{\beta}$ values far from densely represented region.

As mentioned in the summary of section 6, Cespedes methodology, [1], is superior to Pykhtin, [8], in the framework of intensive computations. Even the relaxed version of Pykhtin, denoted by PykhDüll, needs more time to perform one portfolio evaluation if compared to Cesp, under the condition of given DF parametrization.

A drawback of both models is the assumed standard normal distributions for the asset returns (13). This induces less collective downward movements, which are usually observed during financial stress periods. Yet for [1] this assumption can be relaxed and any other copula besides Gaussian can be used to describe portfolio asset return dependence structure. One can choose, for instance, student-t, Clayton or Archimedian copula to model stronger dependence both in stress or economical growth periods. But this gives no rise for comparison studies due to restrictive standard normality assumption in [8].

There are several additional remarks concerning the calibration of methodology [1], which are important when, for instance, comparing [1] with the Basel II, [4], credit risk capital formula. Basel II capital formula is an application of ASRF model and, as a result of DF ≤ 1, is always greater or equal than credit risk capital obtained via [1]. Yet this is an issue of the right choice of risk parameters, which is not the central topic of this thesis. As mentioned before, given a reference portfolio construction together with risk parameters under which the ASRF provides exact results, DF can be rescaled, as performed in [12], such that DF $= 1$ for equivalent portfolios, thus DF ≤ 1 for better diversified (with respect to the reference portfolio) cases and DF ≥ 1 for less diversified portfolios.

A Inter-sector correlation matrices

The often used inter-sector correlation matrices Q_1, Q_2, Q_3 and Q_4 (all in $\mathbb{R}^{K\times K}$) are presented here. The number of sectors is set to $K = 12$. This choice was motivated by a practical example from the industry, yet this does not introduce any specific limitations or superiority to any other $K \in \mathbb{N}$. Like in practice, correlation matrices are calculated based on the data sets. In our case we sampled data from 12 dependent (via a common random factor) normally distributed random variables, whereas the weight of the common factor is increased to obtain relatively higher correlation levels. Only for presentational purpose in this appendix we rounded all matrix entries up to the 4^{th} decimal digit.

Q_1 is the identity matrix. We also call it `nCorr` to emphasize that there is no linear dependence, i.e., no correlation, across sector level performance indices.

Q_2 represents the case of relatively low correlation across sector dynamics, both positively and negatively correlated. We also call it `lCorr`. See table 6. Mean and standard deviation of nondiagonal entries of Q_2 are 0.0166 and

Table 6: Q_2 (`lCorr`)

1	0,05	0,06	0,07	-0,01	0,04	-0,05	-0,03	-0,08	0,05	0,04	-0,02
0,05	1	0,04	0,01	0,04	0,04	0,09	-0,08	0,02	0,11	-0,01	0,05
0,06	0,04	1	-0,06	0,02	-0,05	0,05	0,04	-0,24	-0,03	0,06	-0,1
0,07	0,01	-0,06	1	-0,05	0,04	0,23	-0,01	0,04	0,15	-0,11	0,15
-0,01	0,04	0,02	-0,05	1	0,08	0,16	0	0,09	-0,26	0,27	0,05
0,04	0,04	-0,05	0,04	0,08	1	0,13	-0,03	0,18	-0,05	0,03	0,04
-0,05	0,09	0,05	0,23	0,16	0,13	1	-0,03	0,05	-0,05	0,06	-0,1
-0,03	-0,08	0,04	-0,01	0	-0,03	-0,03	1	-0,1	0,15	-0,03	-0,06
-0,08	0,02	-0,24	0,04	0,09	0,18	0,05	-0,1	1	-0,09	0,09	0,05
0,05	0,11	-0,03	0,15	-0,26	-0,05	-0,05	0,15	-0,09	1	-0,1	-0,01
0,04	-0,01	0,06	-0,11	0,27	0,03	0,06	-0,03	0,09	-0,1	1	0,01
-0,02	0,05	-0,1	0,15	0,05	0,04	-0,1	-0,06	0,05	-0,01	0,01	1

0.0929, respectively.

Matrix Q_3 or, equivalently, `mCorr` was obtained using MSCI data of 12 different industry sector indices. Results are shown in table 7. Mean and standard deviation of nondiagonal entries of Q_3 are 0.4222 and 0.2623, respectively.

Q_4 imitates the case of relatively high positive correlation across sector dynamics. We also call it `hCorr`. Its entries are shown in table 8. Mean and standard deviation of nondiagonal entries of Q_4 are 0.7483 and 0.0622, respectively.

Table 7: Q_3 (`mCorr`)

1	0,57	0,46	0,65	0,33	0,58	0,33	0,59	0,31	0,00	0,23	0,60
0,57	1	0,59	0,77	0,41	0,91	0,78	0,79	0,18	0,06	0,63	0,63
0,46	0,59	1	0,56	0,63	0,60	0,42	0,56	0,13	0,02	0,30	0,55
0,65	0,77	0,56	1	0,48	0,84	0,62	0,76	0,20	0,08	0,54	0,76
0,33	0,41	0,63	0,48	1,00	0,40	0,39	0,39	0,00	-0,06	0,29	0,46
0,58	0,91	0,60	0,84	0,40	1	0,77	0,86	0,21	0,07	0,60	0,69
0,33	0,78	0,42	0,62	0,39	0,77	1	0,58	0,09	0,12	0,74	0,47
0,59	0,79	0,56	0,76	0,39	0,86	0,58	1	0,25	0,09	0,43	0,63
0,31	0,18	0,13	0,20	0,00	0,21	0,09	0,25	1	0,05	0,09	0,21
0,00	0,06	0,02	0,08	-0,06	0,07	0,12	0,09	0,05	1	0,08	0,01
0,23	0,63	0,30	0,54	0,29	0,60	0,74	0,43	0,09	0,08	1	0,50
0,60	0,63	0,55	0,76	0,46	0,69	0,47	0,63	0,21	0,01	0,50	1

Table 8: Q_4 (`hCorr`)

1	0,72	0,63	0,71	0,68	0,79	0,75	0,67	0,68	0,76	0,7	0,71
0,72	1	0,7	0,75	0,78	0,83	0,8	0,69	0,71	0,83	0,7	0,77
0,63	0,7	1	0,71	0,72	0,77	0,71	0,59	0,72	0,72	0,67	0,8
0,71	0,75	0,71	1	0,73	0,81	0,79	0,72	0,72	0,82	0,7	0,79
0,68	0,78	0,72	0,73	1	0,82	0,77	0,65	0,67	0,8	0,66	0,76
0,79	0,83	0,77	0,81	0,82	1	0,87	0,76	0,82	0,91	0,79	0,88
0,75	0,8	0,71	0,79	0,77	0,87	1	0,72	0,74	0,83	0,75	0,84
0,67	0,69	0,59	0,72	0,65	0,76	0,72	1	0,68	0,76	0,69	0,77
0,68	0,71	0,72	0,72	0,67	0,82	0,74	0,68	1	0,77	0,7	0,77
0,76	0,83	0,72	0,82	0,8	0,91	0,83	0,76	0,77	1	0,74	0,86
0,7	0,7	0,67	0,7	0,66	0,79	0,75	0,69	0,7	0,74	1	0,76
0,71	0,77	0,8	0,79	0,76	0,88	0,84	0,77	0,77	0,86	0,76	1

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